TOPOLOGICAL RIGIDIFICATION OF SCHEMES

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ABSTRACT. We show that any of a large class of schemes receives a universal homeomorphism from a reduced scheme that in turn receives no nontrivial universal homeomorphism from any other reduced scheme. This construction serves as a categorical input for the formal inversion of universal homeomorphisms; the result is an ∞-category that embeds as a full subcategory of the h ∞-topos of Voevodsky.

CONTENTS

INTRODUCTION

The idea to treat the collection of *universal homeomorphisms* of schemes as a class of *weak equivalences* seems to have been Grothendieck's: a quarter of a century ago, in his „Brief an Faltings" [Gro97a, (8)], Grothendieck suggested that, for a field K finitely generated over $\mathbf{Q}$, after formally inverting universal homeomorphisms, the functor $X \longmapsto X_{\text{ét}}$ defines a fully faithful embedding of the category of varieties over K into the category of topoi over $(\operatorname{Spec} K)_{\text{ét}}$. More generally, Grothendieck expected that a variant of the conjecture should hold in positive characteristic, but only after passage to the perfect closure of the base field. This conjecture was discussed in the « Esquisse d'un program » [Gro97b, Endnote (3), p. 53], and, in characteristic zero, a version thereof was proved by Voevodsky [Voe90]. (See also [Sti02, §3].)

The contributions of this paper to this story are: (1) to provide a geometric construction — the *topological rigidification* — that yields a model for any quasicompact and quasiseparated scheme up to universal homeomorphism, and (2) to use this construction to invert universal homeomorphisms in the étale topos, yielding an ∞-topos related to the one corresponding to Voevodsky's h topology. In a sequel to this paper, we aim to prove a generalization of Grothendieck's conjecture, which, in effect, identifies a large class of Artin n-stacks of finite type over Spec $\mathbf{Z}$ that can, up to universal homeomorphism, be reconstructed from their étale ∞-topoi.

We describe in detail the contents of this paper and the manner in which its aims are achieved.

Recall that a morphism $f\colon X \longrightarrow Y$ of schemes is said to be a *universal homeomorphism* if any base change of f is a homeomorphism on the underlying topological space. We collect in §1 a number of elementary facts about this remarkable class of morphisms. The various equivalent characterizations of universal homeomorphisms [1.5] yield an effective way of testing whether a morphism is a universal homeomorphism. Thinking of universal homeomorphisms as weak equivalences is justified, in part, by the fact that they satisfy the two-out-of-three axiom [1.6]. Consequently, virtually any question about universal homeomorphisms can be addressed by treating the nilimmersion and schematically dominant cases separately [1.6.1]. We discuss examples and nonexamples of universal homeomorphisms in 1.8 and 1.9.

The study of universal homeomorphisms can be reduced to the study of universal homeomorphisms with good finiteness properties. Any universal homeomorphism can be approximated by a finite universal homeomorphism [2.1]. The study of universal homeomorphisms can be further reduced by means of a factorization of any finite universal homeomorphism as a nilimmersion followed by a universal homeomorphism of finite presentation [2.3]. Universal homeomorphisms of finite presentation can in turn be studied by means of absolute noetherian approximation of the base thanks to 2.4.

The condition that a morphism be a universal homeomorphism can be regarded as *complementary* to the condition that it be étale. The first piece of justification for this manner of thinking comes in §3 from the observation that a universal homeomorphism is étale if and only if it is an isomorphism [3.1].

The second piece of justification for this line of thinking begins with the « équivalence remarquable de catégories » of [EGA$_{\mathrm{IV}}$ IV, 18.1.2]: base change along any nilimmersion $Z \longrightarrow X$ yields an equivalence of categories between the category Ét_X of étale X-schemes and the category Ét_Z of étale Z-schemes; thus the morphism of small étale topoi $Z_{\text{ét}} \longrightarrow X_{\text{ét}}$ is also an equivalence. Grothendieck generalized this result [SGA$_1$, Exp. IX, 4.10] and [SGA$_4$ II, Exp. VIII, 1.1] to all universal homeomorphisms; see also [Ryd07, 5.21]. We give a new, less circuitous, proof (without descent theory) of this « invariance topologique » of the étale topos in §4.

Frobenius morphisms are a good source of examples of universal homeomorphisms. In fact, in §5, we follow work of Kollár [Kol97] to sketch a proof that, in positive characteristic, every finite universal homeomorphism of noetherian schemes is a section up to Frobenius; that is, any universal homeomorphism factors a sufficiently high-powered geometric Frobenius [5.5]. Thus, the geometric Frobenii are cofinal in the category of universal homeomorphisms out of a given noetherian scheme.

A scheme is said to be *topologically rigid* if the only universal homeomorphisms to it are retractions of nil-thickenings. In effect, topologically rigid varieties are those varieties with perfect function fields for which any non-normality is as transverse as possible. We introduce this class of objects in 6.1. Filtered limits of topologically rigid schemes remain topologically rigid [6.6]; this permits us to employ absolute noetherian reduction arguments in the sequel. The condition of topological rigidity has a purely geometric analogue called *seminormality*; these are equivalent in characteristic zero, but in positive characteristic, they differ dramatically, as our examples and nonexamples show [6.11, 6.12].

The inclusion functor from the category of reduced schemes into the category of all schemes admits a right adjoint $Y \longmapsto Y_{\mathrm{red}}$. Similarly, in 7.1 we show that the inclusion functor from the category of topologically rigid schemes into the category of all schemes admits a right adjoint $Y \longmapsto Y_{\mathrm{trig}}$; this construction is called *topological rigidification*. As Y_{red} can be thought of as the best reduced approximation to Y, so Y_{trig} can be thought of as the best topologically rigid approximation to Y.

It is a remarkable consequence of Kollár's result that the topological rigidification of a reduced noetherian $\mathbf{F}_p$-scheme coincides with its perfection; in effect, the limit over all universal homeomorphisms can be replaced by the system of Frobenii [7.4]. This computation of the topological rigidification illustrates two peculiar phenomena in positive characteristic. The first is the ability the Frobenius has to resolve problems of transversality. The second is the price one has to pay for this simplicity: the topological rigidification of a scheme with pleasant finiteness properties is unlikely to retain those good properties. It is for this reason that we are forced to contend with such general categories of schemes. (We note that Voevodsky alludes to the existence of $Y \longmapsto Y_{\mathrm{trig}}$ after [Voe96, 3.2.10], but he does not investigate the construction further, precisely because of its failure to preserve finiteness properties.)

In §8, following the early work of Andreotti–Bombieri [AB69] and the later work of Manaresi [Man80], we sketch proofs of the existence and properties of a relative version of the topological rigidification — the *weak normalization* of a scheme Y relative to a morphism $Z \longrightarrow Y$. This yields a powerful description of the topological rigidification of reduced schemes with locally finitely many irreducible components [8.4.1], and it permits us to prove 8.6.2, which is key for the construction of the final section.

At last, in §9, we invert the universal homeomorphisms in the étale ∞-topos. Our method of doing so makes essential use of the topological rigidification functor. The result is called the *topologically invariant étale ∞-topos*, and it is closely related to the h topology of Voevodsky [Voe96]. Topologically rigid schemes thus form a generating site for the h ∞-topos for which the topology is subcanonical.

Conventions. The language of ∞-categories and ∞-topoi [HTT] is restricted to the final section. There, the assertions have been formulated so that one may simply remove all instances of the phrases "simplicial" and "up to homotopy" as well as the symbol "∞" in order to obtain correct statements and proofs about the (ordinary)

h topos; however, in the sequel to this paper, the use of ∞-topoi will not be so readily avoidable.

We use the Universe Axiom of [SGA$_4$ I, Exp I, §0], and we fix two universes $\mathbf{U} \in \mathbf{V}$. All rings, modules, schemes, etc., will be $\mathbf{U}$-small. We employ the following notational conventions for categories or ∞-categories:

(N.1) Roman characters A, B, C, ..., etc., will denote categories and ∞-categories that are essentially $\mathbf{V}$-small and locally $\mathbf{U}$-small.

(N.2) Bold characters $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, ..., etc., will denote categories and ∞-categories that are locally $\mathbf{V}$-small.

1. REVIEW: UNIVERSAL HOMEOMORPHISMS

Notation 1.0. For a scheme X, denote by $|X|$ the underlying topological space; for a morphism $f \colon X \longrightarrow Y$ of schemes, denote by $|f| \colon |X| \longrightarrow |Y|$ the underlying continuous map. We say that f is *surjective*, *injective*, *closed*, *open*, etc., if and only if $|f|$ is so. Denote by $f^\sharp \colon \mathscr{O}_Y \longrightarrow f_\star \mathscr{O}_X$ the corresponding morphism of sheaves.

Theorem 1.1 ([EGA$_{\mathrm{IV}}$ IV, 18.12.8], [EGA$_{\mathrm{I}}^*$, 3.8.10]). *The following conditions on a morphism $f \colon Y \longrightarrow X$ of schemes are equivalent.*

(1.1.1) The morphism f is integral (entier).

(1.1.2) The morphism f is affine and the $\mathscr{O}_X$-algebra $f_\star \mathscr{O}_Y$ is integral.

(1.1.3) The morphism f is affine and universally closed.

(1.1.4) The morphism f is affine, and for any set S, the induced morphism

$$\mathbf{A}_Y^S := \operatorname{Spec}_Y(\mathscr{O}_Y[S]) \longrightarrow \operatorname{Spec}_X(\mathscr{O}_X[S]) =: \mathbf{A}_X^S$$

is closed.

Lemma 1.2 ([EGA$_{\mathrm{II}}$, 6.1.5(v)]). *Suppose $Z \longrightarrow X$ an integral morphism of schemes, and suppose $Y \longrightarrow X$ a separated morphism of schemes. Then any morphism $Z \longrightarrow Y$ over X is integral as well.*

Definition 1.3. Suppose $f \colon Y \longrightarrow X$ a quasifinite morphism of schemes, and suppose $x \in X$ a point. Following [EGA$_{\mathrm{I}}^*$, I, 6.5.9], we define the *separable rank* of Y over x to be the natural number

$$\operatorname{rk}_x(Y) := \sum_{y \in f^{-1}(x)} [\kappa(y) : \kappa(x)]_{\mathrm{s}}.$$

Theorem 1.4 ([EGA$_{\mathrm{I}}^*$, 3.7.1]). *The following conditions on a morphism $f \colon Y \longrightarrow X$ of schemes are equivalent.*

(1.4.1) The morphism f is universally injective (radiciel).

(1.4.2) The map $|f|$ is injective, and for any point $y \in Y$, if $x = f(y)$, the residue field extension $\kappa(x) \hookrightarrow \kappa(y)$ is purely inseparable.

(1.4.3) For any field k, the induced map $Y(k) \longrightarrow X(k)$ is injective.

(1.4.4) For any field k, there is an algebraically closed extension Ω of k such that the induced map $Y(\Omega) \longrightarrow X(\Omega)$ is injective.

(1.4.5) The diagonal $|\Delta_{Y/X}| \colon |Y| \longrightarrow |Y \times_X Y|$ is surjective.

If, in addition, f is quasifinite, then the following condition may be added.

(1.4.6) For any $x \in |X|$, the separable rank $\operatorname{rk}_x(Y) = 1$.

Corollary 1.4.1. *Suppose $Z \longrightarrow X$ a universally injective morphism of schemes. Then any morphism $Z \longrightarrow Y$ of schemes over X is universally injective as well.*

Theorem 1.5 ([EGA$_{\text{IV}}$ IV, 18.12.11]). *The following conditions on a morphism $f\colon Y \longrightarrow X$ of schemes are equivalent.*

(1.5.1) *The morphism f is a universal homeomorphism.*
(1.5.2) *The morphism f is surjective, universally injective, and universally closed.*
(1.5.3) *The morphism f is surjective, universally injective, and integral.*
(1.5.4) *The morphism f satisfies the following pair of conditions.*
    (1.5.4.1) *For any $x \in |X|$, the fiber $f^{-1}(x) = \{y\}$ such that the residue field extension $\kappa(x) \hookrightarrow \kappa(y)$ is purely inseparable.*
    (1.5.4.2) *For any set S, the induced morphism $\mathbf{A}_Y^S \longrightarrow \mathbf{A}_X^S$ is closed.*

Proposition 1.6. *Suppose X a scheme. The set of universal homeomorphisms over X satisfies the two-out-of-three axiom. More precisely, the following obtains.*

(1.6.1) *If $W \longrightarrow U$ is a universal homeomorphism of X-schemes and $V \longrightarrow U$ is a separated, surjective, and universally injective morphisms of schemes, then any morphism $W \longrightarrow V$ over U is a universal homeomorphism as well.*
(1.6.2) *If $W \longrightarrow V$ a universal homeomorphism of X-schemes, then a morphism $V \longrightarrow U$ of X-schemes is a universal homeomorphism if and only if the composite $W \longrightarrow U$ is so.*

Proof. (1.6.1) follows immediately from 1.4.1 and 1.2.

The second claim is the observation that for any morphism $U' \longrightarrow U$ of X-schemes, then $V \times_U U' \longrightarrow U'$ is a homeomorphism if and only if the composite $W \times_U U' \longrightarrow U'$ is so. □

Corollary 1.6.1. *The class of universal homeomorphisms $f\colon Y \longrightarrow Y'$ admits functorial factorizations into schematically dominant universal homeomorphisms $Y \longrightarrow Y''$ followed by nilimmersions $Y'' \hookrightarrow Y'$.*

Proof. Simply factor f through its schematic image [EGA$_{\text{I}}^{*}$, I 6.10.1]. □

Proposition 1.7. *A schematically dominant universal homeomorphism is an epimorphism in the category of schemes.*

Proof. Any such morphism $f\colon X \longrightarrow Y$ induces a homeomorphism $|f|$ of topological spaces and a monomorphism $f^{\sharp}\colon \mathscr{O}_Y \longrightarrow f_\star \mathscr{O}_X$ of sheaves. □

Example 1.8. (1.8.1) Any nilimmersion is a universal homeomorphism, and any section of a universal homeomorphism is a nilimmersion.
(1.8.2) If $K \hookrightarrow L$ is a purely inseparable field extension, then the morphism $\operatorname{Spec} L \longrightarrow \operatorname{Spec} K$ is a universal homeomorphism. Moreover, for any K-scheme X, the morphism $X \otimes_K L \longrightarrow X$ is a universal homeomorphism.
(1.8.3) Suppose X a reduced scheme whose set of irreducible components is locally finite.[1] Denote by X^ν its normalization. Then the following are equivalent: (1) the natural morphism $X^\nu \longrightarrow X$ is a universal homeomorphism; (2) X is geometrically unibranch [EGA$_{\text{IV}}$ II, 6.15.1]; (3) for every point $x \in X$, the strict henselianization ${}^{\text{sh}}\mathscr{O}_{X,x}$ of the local ring at x is irreducible [EGA$_{\text{IV}}$ IV, 18.8.15(c)].

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[1]This means that every quasicompact open of X contains only finitely many irreducible components. This condition holds, e.g., if the topological space $|X|$ is locally noetherian.

(1.8.4) Suppose k an algebraically closed field of characteristic 2. The subring $k[x^2, xy, y] \subset k[x, y]$ induces a morphism

$$f \colon \operatorname{Spec} k[x, y] \longrightarrow \operatorname{Spec} k[x^2, xy, y]$$

that is finite, surjective, and an isomorphism away from the x-axis. One sees that the ideal (y) lies over the ideal (xy, y); the induced morphism of subschemes

$$\operatorname{Spec} k[x] \longrightarrow \operatorname{Spec} k[x^2]$$

is a bijection inducing an isomorphism of residue fields on every closed point. On the generic point of this subscheme, the field extension is $k(x^2) \subset k(x)$. One concludes that $Y \longrightarrow X$ is a universal homeomorphism.

Counterexample 1.9. (1.9.1) Consider the subring $\mathbf{R}[x, \sqrt{-1}x] \subset \mathbf{C}[x]$. The induced morphism $f \colon \operatorname{Spec} \mathbf{C}[x] \longrightarrow \operatorname{Spec} \mathbf{R}[x, \sqrt{-1}x]$ is a bijection, and away from the origin it is an isomorphism. At the origin, the residue field extension is $\mathbf{R} \subset \mathbf{C}$; hence f is not a universal homeomorphism.

(1.9.2) Suppose k an algebraically closed field of characteristic 0. Suppose X a nodal curve over k, and let $X^\nu \longrightarrow X$ be its normalization, so that over the singular point $x \in X(k)$ there are two points $u, v \in X^\nu(k)$. Remove u, and consider the restriction $X^\nu - \{u\} \longrightarrow X$; this is a bijection on all points that induces an isomorphism on all residue field extensions. It is not, however, an integral morphism.

2. Limits of universal homeomorphisims and approximation

Proposition 2.1. *Suppose X a scheme. Suppose Λ a small filtered category, and suppose $W_* \colon \Lambda^{\mathrm{op}} \longrightarrow (\mathrm{Sch}/X)$ a diagram of X-schemes such that for any object $\alpha \in \Lambda^{\mathrm{op}}$, the structure morphism $p_\alpha \colon W_\alpha \longrightarrow X$ is a universal homeomorphism. Then the natural morphism*

$$p \colon W := \lim_{\alpha \in \Lambda^{\mathrm{op}}} W_\alpha \longrightarrow X$$

is a universal homeomorphism.

Proof. All the bonding morphisms $W_\alpha \longrightarrow W_\beta$ are universal homeomorphisms by 1.6. It follows from [EGA$_{\mathrm{IV}}$ III, 8.3.8(i)] that p is surjective. For any field k, the diagram $W(k) \colon \Lambda^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a diagram of injections, whence for any $\alpha \in \Lambda^{\mathrm{op}}$, the map $W(k) \longrightarrow W_\alpha(k)$ is an injection; thus p is a universal injection. It thus remains to show that p is integral. Since W is a diagram of affine X-schemes, it is enough to observe that the filtered colimit $\operatorname{colim}_{\alpha \in \Lambda} p_{\alpha,\star} \mathscr{O}_{W_\alpha}$ is an integral $\mathscr{O}_X$-algebra. $\square$

Lemma 2.2. *Suppose X a coherent (i.e., quasicompact and quasiseparated) scheme, and suppose $p \colon W \longrightarrow X$ a universal homeomorphism. Then there exists a filtered category Λ and a diagram $W_* \colon \Lambda^{\mathrm{op}} \longrightarrow (W/\mathrm{Sch}/X)$ of X-schemes under W satisfying the following conditions.*

(2.2.1) The natural morphism $W \longrightarrow \lim_{\alpha \in \Lambda^{\mathrm{op}}} W_\alpha$ is an isomorphism.

(2.2.2) For any $\alpha \in \Lambda^{\mathrm{op}}$, the morphism $p_\alpha \colon W_\alpha \longrightarrow X$ is a finite universal homeomorphism.

Proof. Write $\mathscr{A}$ for the quasicoherent $\mathscr{O}_X$-algebra $p_\star \mathscr{O}_W$, so that $W = \operatorname{Spec}_X \mathscr{A}$. By [EGA$_{\mathrm{I}}^*$, I 6.9.15], write $\mathscr{A}$ as a filtered colimit $\operatorname{colim}_{\alpha \in \Lambda} \mathscr{A}_\alpha$ of its sub-$\mathscr{O}_X$-algebras of finite type. For any $\alpha \in \Lambda$, set $W_\alpha := \operatorname{Spec}_X \mathscr{A}_\alpha$. The morphism $W \longrightarrow W_\alpha$

is integral and universally injective; moreover, it is schematically dominant, hence surjective. Hence it is a universal homeomorphism, and by 1.6, so is p_α. $\qquad\square$

Proposition 2.3. *Suppose X a coherent scheme, and suppose $p\colon W \longrightarrow X$ any finite universal homeomorphism. Then there exists a universal homeomorphism $p'\colon W' \longrightarrow X$ of finite presentation and a nilimmersion $W \hookrightarrow W'$ over X.*

Proof. By [Con07, 4.2], p factors as a separated morphism $Z \longrightarrow X$ of finite presentation followed by a closed immersion $i\colon W \hookrightarrow Z$ over X. Set $\mathscr{I} := \ker[\mathscr{O}_Z \longrightarrow i_\star \mathscr{O}_W]$. Let Λ denote the category of quasicoherent subideals $\mathscr{J}_\alpha \subset \mathscr{I}$ of finite type. Now [EGA$_\mathrm{I}^*$, I 6.9.16(iii)] Λ is filtered, and the $\mathscr{O}_Z$-algebra $\mathscr{O}_Z/\mathscr{J}_\alpha$ is of finite presentation for any object $\alpha \in \Lambda$. Write $W_\alpha := \mathrm{Spec}_Z(\mathscr{O}_Z/\mathscr{J}_\alpha)$; hence $\lim_{\alpha \in \Lambda} W_\alpha \cong W$. For any $\alpha \in \Lambda$, $W_\alpha \longrightarrow X$ is surjective, $W \hookrightarrow W_\alpha$ is a closed embedding, and $W_\alpha \hookrightarrow Z$ is a closed embedding. It remains to show that for some $\alpha_0 \in \Lambda$, the morphism p_{α_0} is finite and universally injective. Suppose $\mathscr{V}$ a finite affine open covering of X.

(1) First, for every element $V \in \mathscr{V}$, the scheme $V \times_X W$ is affine; hence there is an $\alpha_{-2} \in \Lambda$ such that for any $V \in \mathscr{V}$, the scheme $V \times_X W_{\alpha_{-2}}$ is affine. Thus for every $\alpha \in (\alpha_{-2}/\Lambda)$, the morphism $W_\alpha \longrightarrow X$ is affine.

(2) Next, for every $V \in \mathscr{V}$, the $\Gamma(V, \mathscr{O}_X)$-algebra $\Gamma(V \times_X W_{\alpha_{-2}}, \mathscr{O}_Z)$ is finitely presented; choose a finite set S_V of generators thereof. The images of these generators in $\Gamma(V \times_X W_\alpha, \mathscr{O}_Z)$ are generators for any $\alpha \in (\alpha_{-2}/\Lambda)$, and since p is finite, the image of each generator $x \in S_V$ in $\Gamma(V \times_X W, \mathscr{O}_Z)$ satisfies a monic polynomial $f_{V,x} \in \Gamma(V, \mathscr{O}_X)[t]$. Hence there exists an $\alpha_{-1} \in (\alpha_{-2}/\Lambda)$ such that the image of $f(V,x)(x)$ vanishes in $\Gamma(V \times_X W_{\alpha_{-1}}, \mathscr{O}_Z)$ for all $V \in \mathscr{V}$ and all $x \in S_V$. Thus $p_{\alpha_{-1}}$ is finite, and for every $\alpha \in (\alpha_{-1}/\Lambda)$, the morphism $p_\alpha\colon W_\alpha \longrightarrow X$ is finite.

(3) Finally, for any $\alpha \in (\alpha_{-1}/\Lambda)$, set

$$Z_\alpha := \{z \in Z \mid \mathrm{rk}_z(W_\alpha) = 1\},$$

where $\mathrm{rk}_z(W_\alpha)$ denotes the separable rank [1.3] of W_α over z. By [EGA$_\mathrm{IV}$ III, 9.7.9], the sets Z_α are constructible subsets of Z. One verifies that

$$Z = \bigcup_{\alpha \in (\alpha_{-1}/\Lambda)} Z_\alpha,$$

a filtered union. Since Z is quasicompact, there is an $\alpha_0 \in \Lambda$ such that $Z = Z_{\alpha_0}$, so for any $\alpha \in (\alpha_0/\Lambda)$, the surjection $W_\alpha \longrightarrow Z$ is universally injective, and for every $\alpha \in (\alpha_0/\Lambda)$, the morphism $p_\alpha\colon W_\alpha \longrightarrow X$ is universally injective. $\qquad\square$

Notation. For the next result, suppose X_0 a scheme, suppose Λ a filtered category, and suppose $\mathscr{A}\colon \Lambda \longrightarrow \mathbf{Alg}(\mathscr{O}_{X_0})$ a diagram of quasicoherent commutative $\mathscr{O}_{X_0}$-algebras. For any $\alpha \in \Lambda$, write $X_\alpha := \mathrm{Spec}_{X_0} \mathscr{A}_\alpha$. Suppose Y_0 and Z_0 two X_0-schemes; for any $\alpha \in \Lambda$, write $Y_\alpha := Y_0 \times_{X_0} X_\alpha$ and $Z_\alpha := Z_0 \times_{X_0} X_\alpha$. Finally, write $X := \lim_{\alpha \in \Lambda^{\mathrm{op}}} X_\alpha$, $Y := \lim_{\alpha \in \Lambda^{\mathrm{op}}} Y_\alpha$, and $Z := \lim_{\alpha \in \Lambda^{\mathrm{op}}} Z_\alpha$.

Theorem 2.4. *Suppose X_0 quasicompact, and suppose Y_0 and Z_0 locally of finite presentation over X_0. Then any universal homeomorphism $Y \longrightarrow Z$ over X descends to a universal homeomorphism $Y_\alpha \longrightarrow Z_\alpha$ over X_α for some $\alpha \in \Lambda$.*

Proof. Apply [EGA$_\mathrm{IV}$ III, 8.10.5(vi, vii, x)]. $\qquad\square$

3. The « propriété fondamentale » of étale morphisms

Proposition 3.1. *A morphism $f\colon Y \longrightarrow X$ of schemes is an étale universal homeomorphism if and only if it is an isomorphism.*

Proof. The sufficiency is obvious. For the necessity, suppose f an étale universal homeomorphism. Then any section of f will be a surjective open immersion [EGA$_\mathrm{IV}$ IV, 17.4.1(b″)], hence an isomorphism. It suffices to construct such a section after a faithfully flat base change. So consider the projection $Y \times_X Y \longrightarrow Y$; the diagonal $\Delta_{Y/X}\colon Y \longrightarrow Y \times_X Y$ is the desired section. $\square$

Corollary 3.1.1 ([EGA$_\mathrm{IV}$ IV, 17.9.1])**.** *A morphism $f\colon Y \longrightarrow X$ of schemes is étale and universally injective if and only if it is an open immersion.*

Proof. The sufficiency is obvious. So suppose f étale and universally injective. Since f is flat and locally of finite presentation, it is universally open [EGA$_\mathrm{IV}$ II, 2.4.6]. It is thus a universal homeomorphism onto its image; one now applies the proposition.

$\square$

Corollary 3.1.2 ([EGA$_\mathrm{IV}$ IV, 17.9.3])**.** *Suppose $f\colon Y \longrightarrow X$ an étale (respectively, étale and separated) morphism of schemes. Then every section of f is an open (resp. open and closed) immersion, and the set $\mathrm{Sect}_X(Y)$ of sections of f is in canonical bijection with the set of open subsets (resp. connected components) $U \subset |Y|$ such that $f|_U\colon U \longrightarrow X$ is a universal homeomorphism and hence an isomorphism.*

4. The « invariance topologique » of the étale topos

Definition 4.1. Suppose E a presheaf of categories on the category Sch of schemes.[2] Then a morphism $Y \longrightarrow X$ of schemes will be called an *E-equivalence* if the induced functor $E_X \longrightarrow E_Y$ is an equivalence of categories; it is called a *universal E-equivalence* if, for any morphism $X' \longrightarrow X$, the induced functor $E_{X'} \longrightarrow E_{X' \times_X Y}$ is an equivalence.

Notation 4.2. For any scheme X, write $\mathrm{\acute{E}t}_X$ for the category of étale X-schemes; then pullback defines a presheaf of categories $\mathrm{\acute{E}t}$ on the category Sch of schemes. Write $\tau_{\leq 1} X_{\mathrm{\acute{e}t}}$ for the ordinary small étale topos of X — i.e., the category of sheaves of sets on $\mathrm{\acute{E}t}_X$; then pullback defines a presheaf of categories $\acute{e}t$ on the category Sch of schemes.

Proposition 4.3. *For any schemes X and Y and any morphism $\phi\colon \tau_{\leq 1} X_{\mathrm{\acute{e}t}} \longrightarrow \tau_{\leq 1} Y_{\mathrm{\acute{e}t}}$ of small étale topoi, there is a unique continuous map $f\colon |X| \longrightarrow |Y|$ such that for any open set $U \subset |Y|$, one has $f^{-1}U = \phi^\star U$.*

Proof. The underlying space of any scheme is sober, so it is enough to observe that such a morphism ϕ induces a homomorphism

$$\mathrm{Zar}_Y \cong \mathrm{Sub}(\tau_{\leq 1} Y_{\mathrm{\acute{e}t}}) \longrightarrow \mathrm{Sub}(\tau_{\leq 1} X_{\mathrm{\acute{e}t}}) \cong \mathrm{Zar}_X$$

of Zariski locales. $\square$

Corollary 4.3.1. *An $\acute{e}t$-equivalence is a homeomorphism.*

Proof. Apply 4.3 to obtain the continuous inverse. $\square$

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[2]By this we mean a pseudofunctor $\mathrm{Sch}^{\mathrm{op}} \longrightarrow \mathscr{C}at_{\mathbf{V}}$ valued in $\mathbf{V}$-small $\mathbf{U}$-categories or, equivalently, a cartesian fibration (with fibers in categories) over Sch.

Proposition 4.4. *For any universal homeomorphism $f\colon Y \longrightarrow X$ of schemes, the induced functor $\text{Ét}_X \longrightarrow \text{Ét}_Y$ is fully faithful.*

Proof. By forming pullbacks, it suffices to show that for any universal homeomorphism $Y \longrightarrow X$ and any étale X-scheme X', the natural map

$$\text{Sect}_X(X') \longrightarrow \text{Sect}_Y(Y'),$$

is a bijection, where $Y' := X' \times_X Y$. Now one may apply 3.1.2: since the projection $X' \times_X Y \longrightarrow X'$ is a universal homeomorphism, the set of open subsets $U \subset X'$ such that the induced morphism $U \longrightarrow X$ is a universal homeomorphism is in canonical bijection with the set of open subsets $V \subset |Y'|$ such that the induced morphism $V \longrightarrow Y$ is a universal homeomorphism. $\square$

Corollary 4.4.1. *Étale morphisms satisfy the unique right lifting property with respect to universal homeomorphisms. That is, for any commutative square*

$$
\begin{array}{ccc}
Z & \longrightarrow & Y \\
\downarrow & & \downarrow \\
W & \longrightarrow & X
\end{array}
$$

of schemes in which $Y \longrightarrow X$ is étale and $Z \longrightarrow W$ is a universal homeomorphism, there exists a unique lift $W \longrightarrow Y$.

Proof. Pulling back along $W \longrightarrow X$, it suffices to consider the case in which $W = X$. By 4.4, for any scheme X and any étale X-schemes Y and Z, the canonical map

$$\text{Mor}_X(X, Y) \longrightarrow \text{Mor}_X(Z, Y) \cong \text{Mor}_X(Z, Y \times_X Y)$$

is a bijection. $\square$

Corollary 4.4.2. *The question of whether an étale Y-scheme descends along a universal homeomorphism $Y \longrightarrow X$ to an étale X-scheme is local on X.*

Proof. Suppose Z an étale Y-scheme, and suppose $\{U_j\}_{j \in J}$ an affine open cover of X; write $\{V_j\}_{j \in J}$ for the pulled back affine open cover of Y. Suppose further that for every $j \in J$, one has an étale U_j-scheme U_j' with an isomorphism

$$\phi_j \colon V_j' := V_j \times_Y Y' \cong V_j \times_{U_j} U_j'.$$

By 4.4, for any $j, k \in J$, there is a unique isomorphism $U_j' \times_{U_j} U_{jk} \cong U_k' \times_{U_k} U_{jk}$ corresponding to the canonical isomorphism $V_j' \times_{V_j} V_{jk} \cong V_k' \times_{V_k} V_{jk}$. These isomorphisms clearly satisfy the cocycle condition, so gluing the U_j' together along these isomorphisms, one obtains an étale X-scheme X' with the property that the ϕ_j glue together to give an isomorphism $Y' \cong X' \times_X Y$. $\square$

Corollary 4.4.3. *Suppose X a coherent scheme. Suppose Λ a filtered category, and suppose $W\colon \Lambda^{\text{op}} \longrightarrow (\text{Sch}/X)$ a diagram of X-schemes such that for any object $\alpha \in \Lambda^{\text{op}}$, the structure morphism $p_\alpha \colon W_\alpha \longrightarrow X$ is a universal homeomorphism that induces an essentially surjective functor $\text{Ét}_X \longrightarrow \text{Ét}_{W_\alpha}$. Then the natural morphism*

$$W := \lim_{\alpha \in \Lambda^{\text{op}}} W_\alpha \longrightarrow X$$

is an Ét-equivalence.

Proof. The functor $\text{Ét}_X \longrightarrow \text{Ét}_W$ is fully faithful by 2.1 and 4.4. Now for any étale morphism $Y \longrightarrow W$, apply [EGA$_{\text{IV}}$ III, 8.8.2] and [EGA$_{\text{IV}}$ IV, 17.7.8] to descend Y to an étale W_α-scheme Y_α for some object $\alpha \in \Lambda$; hence the functor $\text{Ét}_{W_\alpha} \longrightarrow \text{Ét}_W$ — and thus the functor $\text{Ét}_X \longrightarrow \text{Ét}_W$ — is essentially surjective. $\square$

Corollary 4.4.4. *A universal homeomorphism $f \colon Y \longrightarrow X$ is an Ét-equivalence if, for any point $x \in X$, the universal homeomorphism $Y \times_X \operatorname{Spec} \mathscr{O}_{X,x} \longrightarrow \operatorname{Spec} \mathscr{O}_{X,x}$ induces a fully faithful functor $\text{Ét}_{\operatorname{Spec} \mathscr{O}_{X,x}} \longrightarrow \text{Ét}_{Y \times_X \operatorname{Spec} \mathscr{O}_{X,x}}$.*

Proof. The functor $\text{Ét}_X \longrightarrow \text{Ét}_Y$ is fully faithful by 4.4. Write $\operatorname{Spec} \mathscr{O}_{X,x}$ as the cofiltered limit of open affines $\{V_\alpha\}_{\alpha \in \Lambda}$ containing x. Now for any étale Y-scheme Z, the étale $Y \times_X \operatorname{Spec} \mathscr{O}_{X,x}$-scheme $Z \times_X \operatorname{Spec} \mathscr{O}_{X,x}$ descends to an étale $\mathscr{O}_{X,x}$-scheme W_x. Now apply [EGA$_{\text{IV}}$ III, 8.8.2] and [EGA$_{\text{IV}}$ IV, 17.7.8] to descend W_x to an étale V_α-scheme W_α for some object $\alpha \in \Lambda$. Executing this procedure for every point $x \in X$, one obtains an affine open cover $\{U_\alpha\}_{\alpha \in A}$ of X along with an étale U_α-scheme W_α and an isomorphism $W_\alpha \times_X Y \cong Z \times_X U_\alpha$. It now follows from 4.4.2 that there is an étale X-scheme W along with an isomorphism $W \times_X Y \cong Z$. Thus the functor $\text{Ét}_X \longrightarrow \text{Ét}_Y$ is essentially surjective. $\square$

Proposition 4.5. *Nilimmersions are Ét-equivalences.*

Proof. In view of 4.4, our claim is that for any nilimmersion $f \colon Y \longrightarrow X$ of schemes, the induced functor $\text{Ét}_X \longrightarrow \text{Ét}_Y$ is essentially surjective. By 4.4.2, one reduces to the case of $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Write $I = \ker[A \longrightarrow B]$ for the nil ideal defining Y. Now write A as the filtered colimit $\operatorname{colim}_{\alpha \in \Lambda} A_\alpha$ of its subrings of finite type over $\mathbf{Z}$, and set $X_\alpha := \operatorname{Spec} A_\alpha$. Now set

$$B_\alpha := A_\alpha / (I \cap A_\alpha) \quad \text{and} \quad Y_\alpha := \operatorname{Spec} B_\alpha.$$

Thus $Y \cong \lim_{\alpha \in \Lambda^{\mathrm{op}}} Y_\alpha$. For any étale morphism $p \colon Y' \longrightarrow Y$, apply [EGA$_{\text{IV}}$ III, 8.8.2] and [EGA$_{\text{IV}}$ IV, 17.7.8] to descend Y' to an étale Y_α-scheme Y'_α for some object $\alpha \in \Lambda$.

One is thus reduced to the case in which X, Y, and Y' are all noetherian. In this case, $Y \longrightarrow X$ is nilpotent; by induction, one may further assume that it is square zero. The ideal I can then be regarded as a module on Y, and by [Ill71, III 2.1.7], there is an obstruction

$$\varpi(Y'/Y, X) \in \operatorname{Ext}^2(\mathbf{L}_{Y'/Y}, p^\star I)$$

whose vanishing is necessary and sufficient for the existence of a deformation of p. As $\mathbf{L}_{Y'/Y} \simeq 0$, we are done. $\square$

4.6. If Syn denotes the presheaf of categories on Sch of syntomic morphisms $X' \longrightarrow X$, then the argument above shows in fact that any nilimmersion $Y \longrightarrow X$ induces an essentially surjective functor $\operatorname{Syn}_X \longrightarrow \operatorname{Syn}_Y$.

Proposition 4.7. *Suppose X and Y are reduced noetherian local schemes; then any finite universal homeomorphism $f \colon Y \longrightarrow X$ induces an essentially surjective functor $\text{Ét}_X \longrightarrow \text{Ét}_Y$.*

Proof. Write $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ for reduced noetherian local rings $(A, \mathfrak{m}_A, \kappa_A)$ and $(B, \mathfrak{m}_B, \kappa_B)$. It suffices to show that any étale B-scheme Z descends to an étale A-scheme.

When $\dim X = 0$, the ring $A = \kappa_A$ is a field, $B = \kappa_B$ is a finite purely inseparable field extension of A, and $Z \cong \mathrm{Spec}\left(\prod_{i \in I} F_i\right)$ corresponds to a finite product of finite separable field extensions F_i of κ_B. For any $i \in I$, the subfield $E_i \subset F_i$ of elements separable over κ_A gives a defines a finite separable field extension of κ_A such that F_i is purely inseparable over E_i. Each tensor product $E_i \otimes_{\kappa_A} \kappa_B$ is isomorphic to F_i, and the proof is complete in this case.

We now proceed by noetherian induction. Suppose now that the result is known for any X of dimension $\dim X < n$. One may write $Z = Z_0 \sqcup Z_1$, where Z_0 is finite over X, and the fiber of Z_1 over the unique closed point of X is empty. Since Z_1 is étale over the complement over the closed point, the induction hypothesis applies. One is thus reduced to the case in which Z is finite étale over B. In this case, $Z = \mathrm{Spec}\, D$ for some finite étale B-algebra D.

Now consider the completions $\mathfrak{A}$ of A along $\mathfrak{m}_A$, $\mathfrak{B}$ of B along $\mathfrak{m}_B$, and $\mathfrak{D}$ of D along $\mathfrak{m}_B D$. It follows from [EGA$_{\mathrm{IV}}$ IV, 18.3.4] along with the dimension zero case above that $\mathfrak{D}$ descends to a finite étale $\mathfrak{A}$-algebra $\mathfrak{C}$. Now by [EGA$_{\mathrm{III}}$ I, 5.4.4], there is a finite A-algebra C (essentially unique by [EGA$_{\mathrm{III}}$ I, 5.4.1]) whose completion along $\mathfrak{m}_A C$ is $\mathfrak{C}$. The A-algebra C is flat, since both $A \longrightarrow \mathfrak{A}$ and $C \longrightarrow \mathfrak{C}$ are faithfully flat; hence by [EGA$_{\mathrm{IV}}$ IV, 18.4.14], C is a finite étale A-algebra. $\qquad\square$

Theorem 4.8. *Any universal homeomorphism is an* Ét-*equivalence.*

Proof. Suppose $f\colon Y \longrightarrow X$ a universal homeomorphism. The claim is that f is an Ét-equivalence. We employ the earlier results of this section to make a series of reductions to 4.7. (1) By 4.4.2, one may assume that $X = \mathrm{Spec}\, A$ is affine, in which case $Y = \mathrm{Spec}\, B$ is as well. (2) Applying 2.2 and 4.4.3, one may assume that f is of finite type. (3) Applying 2.3 and 4.5, it suffices to assume that f is of finite presentation. (4) Writing A as a filtered colimit of finitely generated $\mathbf{Z}$-algebras, we obtain a description of X as a cofiltered limit of schemes of finite type over $\mathrm{Spec}\,\mathbf{Z}$, and employing 2.4 along with [EGA$_{\mathrm{IV}}$ III, 8.8.2(ii)] and [EGA$_{\mathrm{IV}}$ IV, 17.7.8], one is thus reduced to the case in which A is noetherian. (5) By 4.4.4, one is further reduced to the case in which A is local and noetherian, in which case B is also. (6) By 4.5, one may assume that both A and B are reduced. (7) In view of 4.4, it suffices to show that $\mathrm{Ét}_X \longrightarrow \mathrm{Ét}_Y$ is essentially surjective. This is 4.7. $\qquad\square$

Corollary 4.8.1. *A morphism is a universal homeomorphism if and only if it is a universal ét-equivalence.*

Proof. This is 4.8 combined with 4.3.1. $\qquad\square$

5. Universal homeomorphisms and Frobenii

Notation 5.1. Suppose X an $\mathbf{F}_p$-scheme, and suppose Y an X-scheme. Let $q = p^r$. Denote by $\phi_X^{(q)}\colon X \longrightarrow X$ the absolute q-th power Frobenius [SGA$_5$, XIV=XV §1], and by $Y_{/X}^{(q)}$ the pullback

$$
\begin{array}{ccc}
Y_{/X}^{(q)} & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{\ \phi_X^{(q)}\ } & X
\end{array}
$$

The absolute Frobenius $\phi_Y^{(q)} \colon Y \longrightarrow Y$ thus factors through the *relative* or *geometric* Frobenius

$$\Phi_{Y/X}^{(q)} \colon Y \longrightarrow Y_{/X}^{(q)}.$$

Definition 5.2. An $\mathbf{F}_p$-scheme X is said to be *perfect* if the absolute p-th power Frobenius $\phi_X^{(p)} \colon X \longrightarrow X$ is an isomorphism. Denote by $(\mathrm{Sch}_{\mathrm{perf}}/\mathbf{F}_p)$ the full subcategory of $(\mathrm{Sch}/\mathbf{F}_p)$ comprised of perfect $\mathbf{F}_p$-schemes.

Proposition 5.3. *The inclusion* $(\mathrm{Sch}_{\mathrm{perf}}/\mathbf{F}_p) \hookrightarrow (\mathrm{Sch}/\mathbf{F}_p)$ *admits a right adjoint.*

Proof. Let X_{perf} be the limit of the diagram

$$\cdots \xrightarrow{\ \phi_X\ } X \xrightarrow{\ \phi_X\ } X. \qquad\qquad \square$$

Proposition 5.4 ([SGA$_5$, XIV=XV §1 nᵒ 2, Pr. 2(a)])**.** *The geometric (and hence the absolute) q-th power Frobenius is a universal homeomorphism.*

Corollary 5.4.1 ([SGA$_5$, XIV=XV §1 nᵒ 2, Pr. 2(c)])**.** *Suppose X and Y two $\mathbf{F}_p$-schemes. The following are equivalent for a morphism $f \colon Y \longrightarrow X$ locally of finite presentation.*

(5.4.1.1) The morphism f is unramified (respectively, étale).
(5.4.1.2) The geometric Frobenius $\Phi_{Y/X}^{(p)}$ is unramified (resp., étale).

(5.4.1.3) The geometric Frobenius $\Phi_{Y/X}^{(p)}$ is a monomorphism (resp., isomorphism).

Corollary 5.4.2. *A morphism of perfect $\mathbf{F}_p$-schemes is étale if and only if it is locally of finite presentation.*

Proof. This follows from 5.4.1, since the geometric Frobenius is an isomorphism. $\square$

Proposition 5.5 ((Kollár) [Kol97, 6.6])**.** *Suppose X a quasicompact $\mathbf{F}_p$-scheme, and suppose $f \colon V \longrightarrow U$ a finite universal homeomorphism of noetherian X-schemes. Then there is a $q = p^r$ and a morphism $f^{(-q)} \colon U \longrightarrow V_{/X}^{(q)}$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
 & U & \\
f \nearrow & & \searrow f^{(-q)} \\
V & \xrightarrow{\ \Phi_X^{(q)}\ } & V_{/X}^{(q)}.
\end{array}
$$

Moreover, the morphism $f^{(-q)}$ is functorial in the sense that for any commutative square

$$
\begin{array}{ccc}
V & \xrightarrow{\ f\ } & U \\
\downarrow & & \downarrow \\
T & \xrightarrow{\ g\ } & S
\end{array}
$$

of noetherian X-schemes in which the horizontal morphisms f and g are universal homeomorphisms, for all sufficiently large $q = p^r$ there is a commutative diagram

$$\begin{array}{ccc}
V & \xrightarrow{\Phi_{V/X}^{(q)}} & V_{/X}^{(q)} \\
& f \searrow \quad U \quad \nearrow f^{(-q)} & \\
& \downarrow & \\
& S \searrow \quad g^{(-q)} & \\
T & \xrightarrow[\Phi_{T/X}^{(q)}]{} & T_{/X}^{(q)}.
\end{array}$$

Sketch of proof. The functoriality of $f^{(-q)}$ and the quasicompactness of X permit us to assume X, U, and V affine; write $X = \operatorname{Spec} C$, $U = \operatorname{Spec} A$, and $V = \operatorname{Spec} B$. By 1.6.1 it suffices to assume that $f^\sharp$ is either surjective or injective; i.e., that $f \colon V \longrightarrow U$ is either a nilimmersion or schematically dominant.

Suppose first that $V \longrightarrow U$ is a nilimmersion; let J be the kernel of $f^\sharp$. Since A is noetherian, J is nilpotent; if $v = p^u$ is sufficiently large, then $J^v = 0$. For such q and any element $b \in B$, set $(f^{(-v)})^\sharp(b) = a^v$ for some preimage a of b (it does not matter which). This defines the desired morphism $f^{(-v)} \colon U \longrightarrow V_{/X}^{(v)}$.

Now suppose that $V \longrightarrow U$ is schematically dominant. We construct $f^{(-q)}$ inductively. If U is an Artin scheme, then the result is immediate. One thus finds a $q = p^r$ such that $A \subset B_{/C}^{(q)}A$ is an isomorphism on generic points. (Here $B_{/C}^{(q)}$ is the C-algebra generated by the image of the absolute q-th power Frobenius.) Let I denote the conductor of this extension; then by induction there is a $t = p^s$ such that $(B_{/C}^{(q)}A/I)_{/C}^{(t)} \subset (B/I)$. It follows that $B_{/C}^{(qt)} \subset (B_{/C}^{(q)}A/I)_{/C}^{(t)} \subset (B/I)$, whence one defines the desired morphism $f^{(-qt)} \colon U \longrightarrow V_{/X}^{(qt)}$. $\qquad\square$

6. TOPOLOGICAL RIGIDITY

Notation 6.0. For any scheme X, denote by $(\operatorname{Sch}_{\operatorname{coh}}/X)$ the category of coherent X-schemes.

Definition 6.1. A coherent scheme Y is said to be *topologically rigid* or, in the language of [Ryd07, B.1], *absolutely weakly normal* if any universal homeomorphism $Y' \longrightarrow Y$ with Y' reduced is an isomorphism. Denote by $(\operatorname{Sch}_{\operatorname{trig}}/X)$ the full subcategory of $(\operatorname{Sch}_{\operatorname{coh}}/X)$ spanned by topologically rigid X-schemes; denote by $j \colon (\operatorname{Sch}_{\operatorname{trig}}/X) \hookrightarrow (\operatorname{Sch}_{\operatorname{coh}}/X)$ the inclusion.

Lemma 6.2. *A scheme Y is topologically rigid if and only if any universal homeomorphism $Y' \longrightarrow Y$ admits a section.*

Proof. If Y is topologically rigid, then for any universal homeomorphism $Y' \longrightarrow Y$, the composite $Y'_{\operatorname{red}} \longrightarrow Y$ is an isomorphism. Conversely, a section of a universal homeomorphism $Y' \longrightarrow Y$ is a nilimmersion, hence an isomorphism if Y' is reduced. $\qquad\square$

Lemma 6.3. *A topologically rigid scheme is reduced.*

Proof. If Y is topologically rigid, then $Y_{\operatorname{red}} \longrightarrow Y$ admits a section. $\qquad\square$

Lemma 6.4. *The coproduct of a set $\{U_i\}_{i\in I}$ of schemes is topologically rigid if and only if each scheme U_i is so.*

Proof. This is immediate from the observation that any coproduct of universal homeomorphisms is a universal homeomorphism. $\square$

Lemma 6.5. *A scheme Z is topologically rigid if and only if Z is reduced, and any universal homeomorphism $V \longrightarrow Z$ of finite presentation admits a section.*

Proof. The necessity is obvious. For the sufficiency, suppose Z satisfies the two conditions, and suppose $f\colon W \longrightarrow Z$ a universal homeomorphism, with W reduced; the claim is that f is an isomorphism. Apply 2.2 to write $W = \lim_{\alpha\in\Lambda^{\mathrm{op}}} W_\alpha$ for a cofiltered diagram of coherent schemes over Z under W, where each morphism $W_\alpha \longrightarrow Z$ is finite. For each $\alpha \in M$, one may replace the scheme W_α with the schematic image of the map $W \longrightarrow W_\alpha$, whence we may assume that each W_α is reduced. Now apply 2.3 for each $\alpha \in \Lambda$ to factor $W_\alpha \longrightarrow Z$ as a nilimmersion $W_\alpha \longrightarrow W'_\alpha$ followed by a universal homeomorphism $W'_\alpha \longrightarrow Z$ of finite presentation. Now $W'_\alpha \longrightarrow Z$ admits a section by assumption, and since Z is reduced, this section factors through W_α, whence $W_\alpha \longrightarrow Z$ is an isomorphism. Thus f is a limit of isomorphisms, verifying the claim. $\square$

Proposition 6.6. *Suppose X a coherent scheme, and suppose Λ a filtered category. Then the inclusion $j\colon (\mathrm{Sch}_{\mathrm{trig}}/X) \hookrightarrow (\mathrm{Sch}_{\mathrm{coh}}/X)$ reflects limits of Λ^{op}-diagrams of affine X-schemes.*

Proof. Suppose Z a coherent X-scheme with the property that $Z = \lim_{\alpha\in\Lambda^{\mathrm{op}}} Z_\alpha$, where each Z_α is a topologically rigid affine X-scheme. We employ 6.5: suppose now that $f\colon W \longrightarrow Z$ is a universal homeomorphism of finite presentation. By 2.4 and [EGA$_{\mathrm{IV}}$ III, 8.8.2(ii)], there exists an element $\alpha \in \Lambda$ such that $W \longrightarrow Z$ descends to a universal homeomorphism $W_\alpha \longrightarrow Z_\alpha$. This universal homeomorphism admits a section, since Z_α is topologically rigid, and so $W \longrightarrow Z$ admits a section as well. $\square$

Counterexample 6.7. Observe that j does not commute with finite limits, for even the tensor product of perfect fields may have nilpotents.

6.8. Topological rigidity is related to weak normality [§8], [Man80, Yan85, Kol97, Pic03]. The related geometric notion is that of *seminormality*, which coincides with topological rigidity in characteristic zero. In effect, a scheme is seminormal if and only if all non-normalities are maximally transverse.

Definition 6.9. A universal homeomorphism $f\colon Y' \longrightarrow Y$ of schemes is said to be *arithmetically trivial*[3] if for any point $y' \in Y'$, if $y = f(y')$, then the field extension $\kappa(y) \hookrightarrow \kappa(y')$ is an isomorphism. A coherent scheme Y is said to be *seminormal* [Swa80] if any arithmetically trivial universal homeomorphism $Y' \longrightarrow Y$ with Y' reduced is an isomorphism.

Lemma 6.10. *Any seminormal scheme is reduced.*

Example 6.11. We leave the proofs of these facts as exercises for the amused reader; they will not be used in the sequel.

---

[3]Traverso et al. [Tra70, GT80] use the word "quasi-isomorphism" for such universal homeomorphisms.

(6.11.1) Suppose k a field. Then $\operatorname{Spec} k$ is always seminormal, but it is topologically rigid if and only if k is perfect.

(6.11.2) Suppose X a normal scheme. Then X is seminormal as well. If, for any generic point η of X, the field $\mathscr{O}_{X,\eta} = \kappa(\eta)$ is perfect, then X is topologically rigid.

(6.11.3) Suppose X a reduced scheme whose set of irreducible components is locally finite. Let $\mathscr{R}_X$ denote the quasicoherent $\mathscr{O}_X$-algebra of rational functions on X [EGA$_{\mathrm{I}}^*$, 8.3.3]. Then the following are equivalent (cf. [LV81, 1.4 and 1.7]).

(6.11.3.1) The scheme X is seminormal.

(6.11.3.2) For every point $x \in X$, the local scheme $\operatorname{Spec} \mathscr{O}_{X,x}$ is seminormal.

(6.11.3.3) For every point $x \in X$, and for any integral element $f \in \mathscr{R}_{X,x}$ over $\mathscr{O}_{X,x}$, the conductor of $\mathscr{O}_{X,x}$ in $\mathscr{O}_{X,x}[f]$ is a radical ideal of $\mathscr{O}_{X,x}[f]$.

(6.11.3.4) For every point $x \in X$, if $f \in \mathscr{R}_{X,x}$ is an integral element with the property that both $f^u, f^v \in \mathscr{O}_{X,x}$ for positive relatively prime natural numbers u and v, then $f \in \mathscr{O}_{X,x}$.

(6.11.3.5) For every point $x \in X$, if $f \in \mathscr{R}_{X,x}$ is an integral element with the property that both $f^2, f^3 \in \mathscr{O}_{X,x}$, then $f \in \mathscr{O}_{X,x}$.

(6.11.4) Suppose X a reduced scheme whose set of irreducible components is locally finite; then X is topologically rigid if and only X satisfies the following conditions [Yan83, Th. 1].

(6.11.4.1) The scheme X is seminormal.

(6.11.4.2) For every point $x \in X$ and every integral element $f \in \mathscr{R}_{X,x}^{\mathrm{perf}}$ over $\mathscr{O}_{X,x}$ with the property that there is a prime number p such that $f^p \in \mathscr{O}_{X,x}$ and $pf \in \mathscr{O}_{X,x}$, one has $f \in \mathscr{O}_{X,x}$.

(6.11.5) Suppose k a field. Then a nodal curve (e.g., $y^2 = x^2$ or $y^2 = x^3 + x^2$) over k is seminormal.

(6.11.6) Suppose k a field. Then for any nonnegative integer n, the union of the n coordinate axes in $\mathbf{A}_k^n$ is a seminormal scheme.

Counterexample 6.12. In effect, a scheme can fail to be topologically rigid for two reasons. First, there is the failure of seminormality.

(6.12.1) Suppose k a field. Neither cuspidal curves (e.g., $y^2 = x^3$) nor tacnodal curves (e.g., $y^2 = x^4$) nor ramphoid cusps (e.g., $y^2 = x^5$), etc. ..., are seminormal.

(6.12.2) Suppose k a field. Then for any nonnegative integer n, the scheme comprised of m lines intersecting in one point in $\mathbf{A}_k^n$ is not seminormal if $m > n$.

Beyond the "geometric" considerations of seminormality, there is also the "arithmetic" requirement that each residue field be perfect.

(6.12.3) Suppose k a field of characteristic p; suppose n a positive integer. Then consider the subring

$$A := k[x^{p^n}, x^j y \ : \ 0 \le j < p^n] \subset k[x, y].$$

Since the induced morphism $\mathbf{A}_k^2 \longrightarrow \operatorname{Spec} A$ is a nontrivial universal homeomorphism, it follows immediately that $\operatorname{Spec} A$ is not topologically rigid.

(6.12.4) An $\mathbf{F}_p$-scheme is topologically rigid only if it is perfect. In particular [5.4.2], if X is of finite type over a perfect field k, then X is topologically rigid only if it is *étale* over $\operatorname{Spec} k$.

(6.12.5) If k is a perfect field of characteristic p, then no smooth k-scheme of positive dimension is topologically rigid, but any of them is seminormal.

7. TOPOLOGICAL RIGIDIFICATION

Proposition 7.1. *Suppose X a coherent scheme. The inclusion functor*

$$j\colon (\operatorname{Sch}_{\mathrm{trig}}/X) \hookrightarrow (\operatorname{Sch}_{\mathrm{coh}}/X)$$

admits a right adjoint.

Proof. For any X-scheme Y, denote by (UH/Y) the full subcategory of (Sch/Y) comprised of Y-schemes Z such that the structural morphism $Z \longrightarrow Y$ is a universal homeomorphism. This category contains an essentially ($\mathbf{U}$-)small full cofinal subcategory comprised of the finite universal homeomorphisms $Z \longrightarrow Y$ [2.2]. Let Y_{trig} denote the limit of the diagram $(\mathrm{UH}/Y) \longrightarrow (\operatorname{Sch}/Y)$. This is a cofiltered diagram of affine Y-schemes since finite limits exist in (UH/Y), so 2.1 provides a natural universal homeomorphism $Y_{\mathrm{trig}} \longrightarrow Y$.

The X-scheme Y_{trig} is topologically rigid: indeed, suppose $f\colon Z \longrightarrow Y_{\mathrm{trig}}$ a universal homeomorphism; then the composite $Z \longrightarrow Y$ is a universal homeomorphism, so the projection $\sigma\colon Y_{\mathrm{trig}} \longrightarrow Z$ is a section of f.

Now define a functor in the following manner: assign to any X-scheme Y the object Y_{trig} of (UH/Y). For any morphism $g\colon Y' \longrightarrow Y$ of X-schemes, let

$$g_{\mathrm{trig}}\colon Y'_{\mathrm{trig}} \longrightarrow Y_{\mathrm{trig}}$$

be the composite on the left:

$$
\begin{array}{ccccc}
Y'_{\mathrm{trig}} & \longrightarrow & Y_{\mathrm{trig}} \times_Y Y' & \longrightarrow & Y_{\mathrm{trig}} \\
& \searrow & \downarrow & & \downarrow \\
& & Y' & \xrightarrow{\quad g \quad} & Y.
\end{array}
$$

If Y is an X-scheme, there is a canonical universal homeomorphism $\epsilon\colon Y_{\mathrm{trig}} \longrightarrow Y$; this will be the counit of the adjunction. If Y' is a topologically rigid X-scheme, then the natural morphism $Y'_{\mathrm{trig}} \longrightarrow Y'$ is an isomorphism; the inverse η will be the unit of the adjunction. We now show that the unit and counit give an isomorphism

$$\operatorname{Mor}_X(Y', Y_{\mathrm{trig}}) \cong \operatorname{Mor}_X(Y', Y),$$

thereby completing the proof.

Suppose now Y and Y' two X-schemes; suppose Y' topologically rigid. If $g\colon Y' \longrightarrow Y$ a morphism of X-schemes, one sees easily that the composite

$$Y' \xrightarrow{\eta} Y'_{\mathrm{trig}} \longrightarrow Y_{\mathrm{trig}} \xrightarrow{\epsilon} Y$$

is again g. On the other hand, if $f\colon Y' \longrightarrow Y_{\mathrm{trig}}$ is a morphism of X-schemes, then one forms the corresponding map by $Y' \longrightarrow Y$ composing with ϵ, and one returns

to a morphism $Y' \longrightarrow Y_{\mathrm{trig}}$ by forming the following diagram:

$$
\begin{array}{ccccccc}
Y'_{\mathrm{trig}} & \longrightarrow & Y_{\mathrm{trig}} \times_Y Y' & \xrightarrow{\mathrm{id}\times f} & Y_{\mathrm{trig}} \times_Y Y_{\mathrm{trig}} & \longrightarrow & Y_{\mathrm{trig}} \\
 & {}^{\eta^{-1}}\!\searrow & \downarrow & & \downarrow & & \downarrow{\scriptstyle\epsilon} \\
 & & Y' & \xrightarrow{\;f\;} & Y_{\mathrm{trig}} & \xrightarrow{\;\epsilon\;} & Y.
\end{array}
$$

In order to show that η composed with the long composite on the top is equal to f, it suffices to show that the square

$$
\begin{array}{ccc}
Y'_{\mathrm{trig}} & \longrightarrow & Y_{\mathrm{trig}} \times_Y Y' \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}\times f} \\
Y_{\mathrm{trig}} & \xrightarrow{\;\Delta\;} & Y_{\mathrm{trig}} \times_Y Y_{\mathrm{trig}}
\end{array}
$$

commutes. This is immediate, as Δ is a nilimmersion, and Y'_{trig} is reduced. $\qquad\square$

Definition 7.2. We call the adjoint $(-)_{\mathrm{trig}}$ to j the *topological rigidification*.

Lemma 7.3. *The topological rigidification of any coherent scheme Y is the limit of the natural diagram of Y-schemes indexed by* $(\mathrm{UH}_{\mathrm{red}}/Y) := (\mathrm{UH}/Y) \cap (\mathrm{Sch}_{\mathrm{red}}/Y)$.

Proof. The functor $Z \longmapsto Z_{\mathrm{red}}$ induces a right adjoint to the inclusion $(\mathrm{UH}_{\mathrm{red}}/Y) \hookrightarrow (\mathrm{UH}/Y)$; hence $(\mathrm{UH}_{\mathrm{red}}/Y)$ is cofinal in (UH/Y). $\qquad\square$

Theorem 7.4. *Suppose p a prime number. For any reduced, noetherian $\mathbf{F}_p$-scheme X, the topological rigidification of X is isomorphic to the perfection X_{perf}.*

Proof. Since $X_{\mathrm{perf}} \longrightarrow X$ is a limit of universal homeomorphisms, it is a universal homeomorphism [2.1]. We employ 6.5: suppose $g\colon W \longrightarrow X_{\mathrm{perf}}$ a finite universal homeomorphism of finite presentation. We wish to construct a section of g. By 2.4 and [EGA$_{\mathrm{IV}}$ III, 8.8.2(ii)], W descends to a universal homeomorphism $f\colon V \longrightarrow X$ of finite presentation. Now apply 5.5: there exists a $q = p^r$ and a morphism $f^{(-q)}\colon U \longrightarrow V^{(q)}_{/X}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
 & X & \\
{}^{f}\!\nearrow & & \searrow{\scriptstyle f^{(-q)}} \\
V & \xrightarrow[\Phi^{(q)}_X]{} & V^{(q)}_{/X}.
\end{array}
$$

Now form the pullback of f along the Frobenius $\phi^{(q)}_X$:

$$
\begin{array}{ccc}
V^{(q)}_{/X} & \longrightarrow & V \\
{\scriptstyle f^{(q)}}\downarrow & & \downarrow{\scriptstyle f} \\
X & \xrightarrow[\phi^{(q)}_X]{} & X.
\end{array}
$$

The identities

$$
f^{(q)} \circ \Phi^{(q)}_{/X} = f \quad \text{and} \quad f^{(-q)} \circ f = \Phi^{(q)}_{/X}
$$

together imply that $f^{(q)} \circ f^{(-q)} \circ f = f$, but since f is an epimorphism in the category of schemes [1.7], one deduces that $f^{(-q)}$ is a section of $f^{(q)}$. Since g is the pullback of $f^{(q)}$, it admits a section as well. $\qquad\square$

8. Weak normalization

Definition 8.1. For any morphism $f\colon Y \longrightarrow X$ of schemes, let UH_f denote the category of factorizations

$$Y \longrightarrow X' \longrightarrow X$$

of f with the property that $X' \longrightarrow X$ is a universal homeomorphism and $Y \longrightarrow X'$ is schematically dominant. If UH_f consists of the terminal object alone, then we will say that X is *weakly normal under f* or *weakly normal under Y*. The initial object of UH_f, if it exists, will be called the *weak normalization $X_{\mathrm{wn},f}$ of X under Y* (or the *weak normalization relative to f* if there is possibility for confusion). Thus a scheme X is weakly normal under a morphism $f\colon Y \longrightarrow X$ if and only if the weak normalization exists and coincides with X.

Proposition 8.2. *Suppose $f\colon Y \longrightarrow X$ a morphism of coherent schemes. Suppose $X' \longrightarrow X$ a faithfully flat morphism, and set $Y' := X' \times_X Y$. If X' is weakly normal under Y', then X is weakly normal under Y.*

Proof. Suppose X' weakly normal under Y'. Suppose

$$Y \longrightarrow Z \longrightarrow X$$

a factorization of f with the property that $Z \longrightarrow X$ is a universal homeomorphism. Then the projection $Z' := Z \times_X X' \longrightarrow X'$ is an isomorphism. But by faithfully flat descent, it follows that $Z \longrightarrow X$ is an isomorphism as well. $\square$

Theorem 8.3 ((Andreotti–Bombieri) [AB69, Teorema 2])**.** *Suppose X a reduced, coherent scheme. Suppose $f\colon Y \longrightarrow X$ a dominant (hence schematically dominant [EGA$_{\mathrm{I}}^*$, 5.4.3]), coherent morphism of schemes. Then the weak normalization of X under Y exists.*

Sketch of proof. Write $\mathscr{A}$ for the integral closure of $\mathscr{O}_X$ in $f_\star \mathscr{O}_Y$. Since the affination $\mathrm{Spec}_X \mathscr{A}$ of Y over X is dominant, and since universal homeomorphisms are affine, one may assume that Y is affine over X.

For any point $x \in X$, let $p = \operatorname{char} \kappa(x)$, and set

$$^{\mathrm{wn},f}\mathscr{O}_{X,x} := \{a \in \mathscr{A}_x \mid \exists m \in \mathbf{N},\ a^{p^m} \in f_x^\sharp(\mathscr{O}_{X,x}) + \mathscr{J}(\mathscr{A}_x)\},$$

where $\mathscr{J}$ denotes the Jacobson radical. Now define, for any open set U of X,

$$^{\mathrm{wn},f}\mathscr{O}_X(U) := \{a \in \mathscr{A}(U) \mid \forall x \in U,\ a_x \in\ ^{\mathrm{wn},f}\mathscr{O}_{X,x}\}.$$

One verifies that this defines a quasicoherent $\mathscr{O}_X$-subalgebra $^{\mathrm{wn},f}\mathscr{O}_X$ of $\mathscr{A}$ [AB69, Proposizione 4].

Now set $X_{\mathrm{wn},f} := \mathrm{Spec}_X({}^{\mathrm{wn},f}\mathscr{O}_X)$. The first claim is that the morphism $^{\mathrm{wn}}f\colon X_{\mathrm{wn},f} \longrightarrow X$ is a universal homeomorphism. Indeed, it is a straightforward matter to verify that $^{\mathrm{wn},f}\mathscr{O}_X$ is an integral $\mathscr{O}_X$-algebra and that $^{\mathrm{wn}}f$ is surjective. To show that $^{\mathrm{wn}}f$ is universally injective, in effect, one shows that, for any point $x \in X$, the ring $^{\mathrm{wn},f}\mathscr{O}_{X,x}$ is a local ring whose maximal ideal is the Jacobson radical $\mathscr{J}(\mathscr{A}_x)$. To verify that $^{\mathrm{wn}}f$ is universally injective at x, one wishes to show that the field extension

$$\kappa(x) \longrightarrow ({}^{\mathrm{wn},f}\mathscr{O}_{X,x}/\mathscr{J}(\mathscr{A}_x))$$

is purely inseparable. This follows from the observation that an element $a \in ({}^{\mathrm{wn},f}\mathscr{O}_{X,x}/\mathscr{J}(\mathscr{A}_x))$ only if for some nonnegative integer m, one has $a^{p^m} \in \kappa(x)$.

Now suppose $\mathscr{B}$ a quasicoherent $\mathscr{O}_X$-subalgebra of $\mathscr{A}$ with the property that the morphism $\operatorname{Spec}_X \mathscr{B} \longrightarrow X$ is a universal homeomorphism. Then for any point $x \in X$, it is easy to see that $\mathscr{B}_x$ is contained in the $\mathscr{O}_{X,x}$-subalgebra of $\mathscr{A}_x$ generated by $\mathscr{J}(\mathscr{A}_x)$. From this it follows that $X_{\mathrm{wn},f}$ has the desired universal property. $\square$

Proposition 8.4. *Suppose $Y \longrightarrow X$ a dominant morphism of reduced, coherent schemes. Then the weak normalization of X under the composite $Y_{\mathrm{trig}} \longrightarrow Y \longrightarrow X$ coincides with the topological rigidification of X.*

Proof. Write f for the composite $Y_{\mathrm{trig}} \longrightarrow Y \longrightarrow X$. Then $X_{\mathrm{wn},f} \longrightarrow X$ is a universal homeomorphism, and for any universal homeomorphism $g \colon X' \longrightarrow X_{\mathrm{wn},f}$, the morphism $Y_{\mathrm{trig}} \longrightarrow Y$ factors through $X' \times_X Y$ and thus through X' itself, whence g admits a section. Hence $X_{\mathrm{wn},f}$ is the topological rigidification. $\square$

Notation. For the next result, we use the following notations. Suppose X a reduced scheme whose set of irreducible components is locally finite. Let $X^{(0)}$ be the set of generic points of irreducible components of X, and set

$$\Xi := \coprod_{\eta \in X^{(0)}} \operatorname{Spec}\left(\kappa(\eta)^{\mathrm{perf}}\right);$$

then the natural morphism $\pi \colon \Xi \longrightarrow X$ is dominant.

Corollary 8.4.1. *The weak normalization of X under $\pi \colon \Xi \longrightarrow X$ is the topological rigidification of X.*

Proof. Apply 8.4, noting that by 6.4, Ξ is topologically rigid. $\square$

8.5. Suppose X a reduced scheme whose set of irreducible components is finite, or, respectively, an affine scheme. Then one may form "the" *absolute integral closure* $\overline{X}$ of X [Art71] or, respectively, "the" *total integral closure* $\overline{X}$ of X [Eno68, Hoc70]. One can show that the topological rigidification of X is isomorphic to its weak normalization under $\overline{X} \longrightarrow X$. This construction of X_{trig} is less satisfying, however, since it only recovers the noncanonical isomorphism type.

Proposition 8.6 ((Manaresi), [Man80, I.6]). *Suppose A and A' reduced rings, and suppose $A \hookrightarrow A'$ an integral extension. Then the weak normalization of $\operatorname{Spec} A$ under $\operatorname{Spec} A'$ is $\operatorname{Spec} A''$, where A'' is the equalizer*

$$A'' \longrightarrow A' \rightrightarrows (A' \otimes_A A')_{\mathrm{red}}$$

in the category of rings.

Sketch of proof. Write $^{A'/\mathrm{wn}}A$ for the ring such that $\operatorname{Spec}(^{A'/\mathrm{wn}}A)$ is weak normalization of $\operatorname{Spec} A$ under $\operatorname{Spec} A'$. One verifies directly that $\operatorname{Spec} A'' \longrightarrow \operatorname{Spec} A$ is radicial and surjective, hence a universal homeomorphism, so we have the inclusion $A'' \subset {}^{A'/\mathrm{wn}}A$. On the other hand, every element $a \in {}^{A'/\mathrm{wn}}A$ has the property that $a \otimes 1 - 1 \otimes a$ is a nilpotent element of $A' \otimes_A A'$, so we also have the inclusion $A'' \supset {}^{A'/\mathrm{wn}}A$. $\square$

Corollary 8.6.1 ([Ryd07, B.6(i)]). *Suppose*

$$
\begin{array}{ccc}
Y' & \xrightarrow{\ h'\ } & X' \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
Y & \xrightarrow{\ h\ } & X
\end{array}
$$

a pullback square of reduced coherent schemes. Suppose also that f and g are dominant, and that h is étale. Then the square

$$
\begin{array}{ccc}
Y_{g/\mathrm{wn}} & \longrightarrow & X_{f/\mathrm{wn}} \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\ h\ } & X
\end{array}
$$

is a pullback square.

Proof. The question is local, so assume X and Y affine. Form the normalizations $X_{f/\mathrm{n}}$ and $Y_{g/\mathrm{n}}$ of X and Y under W; by [EGA$_{\mathrm{IV}}$ IV, 18.12.15], one has $Y_{g/\mathrm{n}} \cong X_{f/\mathrm{n}} \times_X Y$, whence we may assume f and g integral. Write $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $X' = \operatorname{Spec} A'$, and $Y' = \operatorname{Spec} B'$. Hence $X_{f/\mathrm{wn}} = \operatorname{Spec} A''$ for A'' the equalizer in 8.6. By the flatness of $A \hookrightarrow B$ combined with [EGA$_{\mathrm{IV}}$ IV, 17.5.7], the diagram

$$
A'' \otimes_A B \longrightarrow B' \rightrightarrows (B' \otimes_{A'} B')_{\mathrm{red}}
$$

is an equalizer as well. $\qquad\square$

Corollary 8.6.2 ([Ryd07, B.6(ii)]). *Suppose X and Y reduced, coherent schemes. Suppose $p\colon Y \longrightarrow X$ an étale morphism. Then the following square is a pullback square*

$$
\begin{array}{ccc}
Y_{\mathrm{trig}} & \longrightarrow & Y \\
{\scriptstyle p_{\mathrm{trig}}}\downarrow & & \downarrow{\scriptstyle p} \\
X_{\mathrm{trig}} & \longrightarrow & X.
\end{array}
$$

Proof. By absolute noetherian approximation [TT90, C.9] combined with 6.6, [EGA$_{\mathrm{IV}}$ III, 8.8.2], [EGA$_{\mathrm{IV}}$ IV, 17.7.8], and [EGA$_{\mathrm{IV}}$ III, 8.7.1], one reduces to the case in which X is of finite type over $\operatorname{Spec}\mathbf{Z}$. By 8.4 and 8.6.1, it is enough to note that Ξ is a topologically rigid scheme with a dominant morphism $\Xi \longrightarrow X$ such that $\Xi \times_X Y$ is topologically rigid and $\Xi \times_X Y \longrightarrow Y$ is dominant. $\qquad\square$

Counterexample 8.7. Suppose X, Y, and Y' reduced coherent schemes; suppose $Y \longrightarrow X$ an étale morphism; and suppose $Y' \longrightarrow Y$ a universal homeomorphism. In general there is not a universal homeomorphism $X' \longrightarrow X$ and an identification of $Y' \cong X' \times_X Y$. To see this, let $X = \operatorname{Spec} k$ for some imperfect field k, $Y = \operatorname{Spec}(k \times k)$, and $Y' = \operatorname{Spec}(k^{\mathrm{perf}} \times k)$.

9. The h site and the étale site of topologically rigid schemes

9.0. In this section, the language of ∞-categories and ∞-topoi [HTT] will be used systematically. In particular, we do not distinguish notationally an ordinary category from its nerve.

Notation 9.1. Suppose X a coherent scheme. The inclusion

$$j \colon (\mathrm{Sch}_{\mathrm{trig}}/X) \hookrightarrow (\mathrm{Sch}_{\mathrm{coh}}/X)$$

induces a string of adjoints (left to right) $(j_\sharp, j^\star, j_\star, j^\natural)$ between the ∞-categories of **V**-small presheaves of simplicial sets on $(\mathrm{Sch}_{\mathrm{coh}}/X)$ and on $(\mathrm{Sch}_{\mathrm{trig}}/X)$, where for any presheaf E of simplicial sets on $(\mathrm{Sch}_{\mathrm{coh}}/X)$ and any presheaf Θ on $(\mathrm{Sch}_{\mathrm{trig}}/X)$, one has

$$j_\star E = E((-)_{\mathrm{trig}}) \qquad \text{and} \qquad j^\star \Theta = \Theta(j(-)),$$

$j_\sharp E$ is the usual left Kan extension of F along j, and $j^\natural \Theta$ is the right Kan extension of G along $Y \longmapsto Y_{\mathrm{trig}}$.[4]

Definition 9.2. The ∞-topos of sheaves of **V**-small simplicial sets [HTT, 6.2.2.6] on the category $(\mathrm{Sch}_{\mathrm{coh}}/X)$ equipped with the étale topology will be called the *large étale ∞-topos* and will be denoted

$$(\mathbf{\acute{E}t}/X) := \mathbf{Shv}_{\mathrm{\acute{e}t}}(\mathrm{Sch}_{\mathrm{coh}}/X).$$

(This is not a **V**-small category; it is rather a topos in the universe **V** [HTT, 6.3.5.17].)

The étale topology can be restricted to $(\mathrm{Sch}_{\mathrm{trig}}/X)$; the resulting ∞-topos of sheaves of **V**-small simplicial sets [HTT, 6.2.2.6] will be called the *topologically invariant étale ∞-topos* (also an ∞-topos in **V**) and will be denoted

$$(\mathbf{\acute{E}t}_{\mathrm{trig}}/X) := \mathbf{Shv}_{\mathrm{\acute{e}t}}(\mathrm{Sch}_{\mathrm{trig}}/X).$$

Lemma 9.3. *Suppose X a coherent scheme. Then the functor $Y \longmapsto Y_{\mathrm{trig}}$ induces a morphism of sites*

$$j \colon (\mathrm{Sch}_{\mathrm{trig}}/X)_{\mathrm{\acute{e}t}} \longrightarrow (\mathrm{Sch}_{\mathrm{coh}}/X)_{\mathrm{\acute{e}t}}$$

and thus a geometric morphism of ∞-topoi

$$j \colon (\mathbf{\acute{E}t}_{\mathrm{trig}}/X) \longrightarrow (\mathbf{\acute{E}t}/X).$$

Proof. By 8.6.2 if $\{V_i \longrightarrow U\}_{i \in I}$ is an étale covering of an X-scheme U, then each morphism $V_{i,\mathrm{trig}} \longrightarrow U_{\mathrm{trig}}$ is étale, and since $U_{\mathrm{trig}} \longrightarrow U$ is a universal homeomorphism, $\coprod_{i \in I} V_{i,\mathrm{trig}} \longrightarrow U_{\mathrm{trig}}$ is faithfully flat. Thus j is continuous.

For any étale sheaf H over X, the presheaf $j^\star H$ is already a topologically invariant sheaf. Hence $j^\star$ admits a further left adjoint $j_!$, which is the composite of the functor $j_\sharp$ with étale sheafification. In particular, $j^\star$ is left exact. $\qquad\square$

Definition 9.4. For any coherent scheme X, we will say that a sheaf $E \in (\mathbf{\acute{E}t}/X)$ is *topologically invariant* if, for any universal homeomoprhism $Y' \longrightarrow Y$ over X, the induced map $E(Y) \longrightarrow E(Y')$ is an equivalence.

Proposition 9.5. *Suppose X a coherent scheme. Then the geometric morphism j is an embedding of topoi that identifies the topologically invariant étale ∞-topos with the full subcategory of $(\mathbf{\acute{E}t}/X)$ spanned by the topologically invariant sheaves.*

---

[4]These notations are compatible with those of Giraud [Gir71, Chap. 0, §3] if one defines j^{-1} as the functor $Y \longmapsto Y_{\mathrm{trig}}$.

Proof. It is immediate that $j^\star j_\star \simeq \mathrm{id}$, so j is indeed an embedding of ∞-topoi; moreover, for any étale sheaf E, the natural morphism $E \longrightarrow j_\star j^\star E$ is an equivalence if and only if, for any X-scheme Y, the induced map $E(Y) \longrightarrow E(Y_{\mathrm{trig}})$ is an equivalence. $\square$

Corollary 9.5.1. *For any morphism $f \colon X \longrightarrow Y$ of coherent schemes, there is a morphism of topoi $f \colon (\acute{\mathbf{E}}\mathbf{t}_{\mathrm{trig}}/X) \longrightarrow (\acute{\mathbf{E}}\mathbf{t}_{\mathrm{trig}}/Y)$ such that the following diagrams commute up to homotopy:*

$$
\begin{array}{ccc}
(\acute{\mathbf{E}}\mathbf{t}_{\mathrm{trig}}/X) & \xrightarrow{\ f_\star\ } & (\acute{\mathbf{E}}\mathbf{t}_{\mathrm{trig}}/Y) \\
{\scriptstyle j_\star}\Big\uparrow & & \Big\uparrow{\scriptstyle j_\star} \\
(\acute{\mathbf{E}}\mathbf{t}/X) & \xrightarrow{\ f_\star\ } & (\acute{\mathbf{E}}\mathbf{t}/Y)
\end{array}
\qquad and \qquad
\begin{array}{ccc}
(\acute{\mathbf{E}}\mathbf{t}_{\mathrm{trig}}/Y) & \xrightarrow{\ f^\star\ } & (\acute{\mathbf{E}}\mathbf{t}_{\mathrm{trig}}/X) \\
{\scriptstyle j_\star}\Big\uparrow & & \Big\uparrow{\scriptstyle j_\star} \\
(\acute{\mathbf{E}}\mathbf{t}/Y) & \xrightarrow{\ f^\star\ } & (\acute{\mathbf{E}}\mathbf{t}/X).
\end{array}
$$

Proof. Using 9.5 to regard the topologically invariant étale ∞-topos as a full subcategory of the étale ∞-topos, the claims are: (1) that for any topologically invariant sheaf $E \in (\acute{\mathbf{E}}\mathbf{t}/X)$, the sheaf $f_\star E$ is topologically invariant, and (2) that for any topologically invariant sheaf $H \in (\acute{\mathbf{E}}\mathbf{t}/Y)$, the sheaf $f^\star E$ is topologically invariant. The first claim follows from the fact that pullbacks of universal homeomorphisms are universal homeomorphisms, and the second is obvious. $\square$

Lemma 9.6. *Suppose X a coherent scheme. The inverse image functor*

$$j^\star \colon (\acute{\mathbf{E}}\mathbf{t}/X) \longrightarrow (\acute{\mathbf{E}}\mathbf{t}_{\mathrm{trig}}/X)$$

sends representables to representables; in particular, one has an isomorphism of topologically invariant sheaves $i^\star Y \cong Y_{\mathrm{trig}}$ for any coherent X-scheme Y.

Proof. Suppose Y an X-scheme. Then for any h sheaf E, one has

$$\mathrm{Map}_{\acute{\mathbf{E}}\mathbf{t}_{\mathrm{trig}}}(Y_{\mathrm{trig}}, E) \simeq E(Y_{\mathrm{trig}}) = j_\star E(Y) \simeq \mathrm{Map}_{\acute{\mathbf{E}}\mathbf{t}}(Y, j_\star E) \simeq \mathrm{Map}_{\acute{\mathbf{E}}\mathbf{t}_{\mathrm{trig}}}(j^\star Y, E),$$

so the result follows from Yoneda. $\square$

9.7. We have constructed a string of adjoints $(j_!, j^\star, j_\star)$ relating the étale and the topologically invariant étale ∞-topoi. Perhaps contrary to expectation, the functor $j_!$ does not provide an identification of the h ∞-topos $(\acute{\mathbf{E}}\mathbf{t}_{\mathrm{trig}}/X)$ with a slice ∞-topos $((\acute{\mathbf{E}}\mathbf{t}/X)/Y)$; that is, j is not an étale morphism of topoi in the sense of [HTT, 6.3.5]. Nevertheless, the functor $j_!$ has a clear geometric meaning, since it preserves representables; in particular, for any étale sheaf E, one may write H_{trig} for $j_! j^\star E$, since the endofunctor $j_! j^\star$ is the left Kan extension of $Y \longmapsto Y_{\mathrm{trig}}$ along the Yoneda embedding. This permits one to say when a large étale sheaf (e.g., a Deligne–Mumford n-stack) is topologically rigid.

Definition 9.8. A large étale sheaf $H \in (\acute{\mathbf{E}}\mathbf{t}/X)$ over a scheme X is said to be *topologically rigid* if the natural morphism $E_{\mathrm{trig}} \longrightarrow E$ is an equivalence.

Proposition 9.9. *For any coherent scheme X, the functor $j_! \colon (\acute{\mathbf{E}}\mathbf{t}_{\mathrm{trig}}/X) \longrightarrow (\acute{\mathbf{E}}\mathbf{t}/X)$ is a fully faithful functor that identifies $(\acute{\mathbf{E}}\mathbf{t}_{\mathrm{trig}}/X)$ with the full subcategory of $(\acute{\mathbf{E}}\mathbf{t}/X)$ spanned by the topologically rigid sheaves.*

Proof. The claim is that the counit id $\longrightarrow j^\star j_!$ is an equivalence. Since both $j^\star$ and $j_!$ commute with colimits, it is enough to show that for any topologically rigid X-scheme Y, the morphism $Y \longrightarrow j^\star j_! Y$ is an equivalence. Since the étale topology is subcanonical, $j_! Y$ is isomorphic to jY, and the result follows from 9.6. $\square$

Definition 9.10. The topology induced on $(\mathrm{Sch}_{\mathrm{coh}}/X)$ by the topologically invariant étale ∞-topos [HTT, 6.2.4.2] is called the *topologically invariant étale topology*. In other words, the topologically invariant étale topology is the topology generated by the étale coverings and universal homeomorphisms.

9.11. The h topology of Voevodsky and Suslin–Voevodsky [Voe96, SV96] is in fact strictly finer than the topologically invariant étale topology, but the sheafifications of representables with respect to these two topologies coincide. To formulate this, fix an excellent scheme X, and denote by $(\mathrm{Sch}^{\sim,\mathrm{trig}}/X)$ the full subcategory of the ∞-topos $(\mathbf{Ét}_{\mathrm{trig}}/X)$ spanned by those sheaves that are represented by the topological rigidification of a scheme of finite type over X, and denote by $(\mathrm{Sch}^{\sim,\mathrm{h}}/X)$ the full subcategory of the ∞-topos $\mathbf{Shv}_{\mathrm{h}}(\mathrm{Sch}_{\mathrm{ft}}/X)$ spanned by those sheaves obtained as the sheafification of representable presheaves.

Theorem 9.12. *The ∞-categories $(\mathrm{Sch}^{\sim,\mathrm{trig}}/X)$ and $\mathbf{Shv}_{\mathrm{h}}(\mathrm{Sch}_{\mathrm{ft}}/X)$ are canonically equivalent.*

Proof. This is now a formal consequence of [Voe96, 3.2.9]. $\square$

Corollary 9.12.1. *Topologically rigid schemes over X form a generating site for the h ∞-topos for which the topology is subcanonical.*

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