DELIGNE COHOMOLOGY

1. Pure Hodge structures

The singular cohomology of smooth projective varieties over $\mathbf{C}$ naturally takes values not in the category of abelian groups, but in the category of *Hodge structures*.

1.1. **Definition.** Denote by $\mathbf{S}$ the real algebraic group $\mathbf{C}^{\times}$; that is, $\mathbf{S}$ is the Weil restriction of the complex algebraic group $\mathbf{G}_m$ to $\mathbf{R} \subset \mathbf{C}$. Denote by $w\colon \mathbf{G}_m \longrightarrow \mathbf{S}$ the canonical morphism that on real points is the inclusion $\mathbf{R}^{\times} \hookrightarrow \mathbf{C}^{\times}$.

Now a *Hodge structure* is a finite rank abelian group $\mathrm{H}_{\mathbf{Z}}$ along with an action σ of the real algebraic group $\mathbf{S}$ on $\mathrm{H}_{\mathbf{R}} := \mathrm{H}_{\mathbf{Z}} \otimes \mathbf{R}$. We will say that $(\mathrm{H}_{\mathbf{Z}}, \sigma)$ is *pure of weight k* if the action of $\sigma w(t)$ on $\mathrm{H}_{\mathbf{R}}$ is action by t^k for any $t \in \mathbf{R}^{\times}$.

1.2. An action of $\mathbf{S}(\mathbf{C}) \cong \mathbf{C}^{\times} \times \mathbf{C}^{\times}$ on $\mathrm{H}_{\mathbf{C}} := \mathrm{H}_{\mathbf{Z}} \otimes \mathbf{C}$ is specified by the decomposition

$$\mathrm{H}_{\mathbf{C}} = \bigoplus_{p,q \in \mathbf{Z}} \mathrm{H}^{p,q}, \quad \mathrm{H}^{p,q} = \{x \in \mathrm{H}_{\mathbf{C}} \mid \forall (u,v) \in \mathbf{S}(\mathbf{C}),\ (u,v)x = u^{-p}v^{-q}x\}.$$

This representation is real just in case $\overline{\mathrm{H}^{q,p}} = \mathrm{H}^{p,q}$. Hence a Hodge structure can be defined as such a decomposition. This Hodge structure is of weight k if and only if $\mathrm{H}^{p,q} = 0$ unless $p + q = k$.

This decomposition also specifies a filtration of $\mathrm{H}_{\mathbf{C}}$, called the *Hodge filtration*:

$$\cdots F^{p+1}\mathrm{H} \subset F^{p}\mathrm{H} \subset \cdots \subset \mathrm{H}_{\mathbf{C}},$$

given by

$$F^{p}\mathrm{H} := \bigoplus_{r \geq p, s \in \mathbf{Z}} \mathrm{H}^{r,s}.$$

The Hodge structure H is pure of weight k just in case $F^{q}\mathrm{H} \cap \overline{F^{p}\mathrm{H}} = 0$ whenever $p + q = k + 1$.

1.3. **Example.** Define a Hodge structure

$$\mathbf{Z}(1) := (2\pi\sqrt{-1})\mathbf{Z} = \ker\left[\exp\colon \mathbf{C} \longrightarrow \mathbf{C}^{\times}\right] \quad \text{with} \quad \mathbf{Z}(1)^{-1,-1} = \mathbf{Z}(1);$$

this is called the *Tate Hodge structure*, pure of weight -2. Its tensor powers are Hodge structures

$$\mathbf{Z}(n) := (2\pi\sqrt{-1})^{n}\mathbf{Z} \subset \mathbf{C} \quad \text{with} \quad \mathbf{Z}(n)^{-n,-n} = \mathbf{Z}(n);$$

these are pure of weight $-2n$.

1.4. **Example.** If X is a compact Kähler manifold, then the holomorphic Poincaré lemma guarantees a quasi-isomorphism $\Omega_X := \Omega_X^\bullet \simeq \mathbf{C}_X$. Now the "foolish" filtration

$$\cdots \longrightarrow \Omega_X^{\geq n} \longrightarrow \Omega_X^{\geq n-1} \longrightarrow \cdots \longrightarrow \Omega_X$$

gives rise to a spectral sequence

$$E_1^{p,q} = \mathrm{H}^q(X, \Omega_X^p) \Longrightarrow \mathrm{H}^{p+q}(X, \mathbf{C})$$

whose abutment is the Hodge filtration on $\mathrm{H}^*(X, \mathbf{C})$. From Hodge theory, we know that this spectral sequence degenerates, whence we obtain a decomposition

$$\mathrm{H}^k(X, \mathbf{C}) = \bigoplus_{p+q=k} \mathrm{H}^q(X, \Omega^p).$$

Moreover, one has $\overline{\mathrm{H}^p(X, \Omega^q)} = \mathrm{H}^q(X, \Omega^p)$. Thus the singular cohomology $\mathrm{H}^k(X, \mathbf{Z})$ is a Hodge structure pure of weight k.

This is all neatly summarized in the statement that the singular cohomology of compact Kähler manifolds (and thus of smooth projective varieties over $\mathbf{C}$) is "really" valued in the category of Hodge structures.

2. Mixed Hodge structures

One does not find a pure Hodge structure of weight k on the singular cohomology $\mathrm{H}^k(U, \mathbf{Z})$ of a general quasiprojective variety U over $\mathbf{C}$. Instead, the weights are mixed.

2.1. **Definition.** A *mixed Hodge structure* is a filtered object

$$\cdots \subset W_{k-1}\mathrm{H}_\mathrm{Z} \subset W_k\mathrm{H}_\mathrm{Z} \subset \cdots \subset \mathrm{H}_\mathrm{Z}$$

in the category of Hodge structures whose k-th graded piece

$$\mathrm{gr}_k^W \mathrm{H}_\mathrm{Z} := W_k\mathrm{H}_\mathrm{Z}/W_{k-1}\mathrm{H}_\mathrm{Z}$$

is pure of weight k.

2.2. **Example.** If H is a mixed Hodge structure, then for any integer m we can define the m-th *Tate twist* of H as $\mathrm{H}(m) := \mathrm{H} \otimes \mathbf{Z}(m)$; in particular,

$$\mathrm{H}(m)_\mathrm{Z} = (2\pi\sqrt{-1})m\mathrm{H}_\mathrm{Z}, \quad W_k\mathrm{H}(m) = W_{k+m}\mathrm{H}, \quad F^p\mathrm{H}(m) = F^{p+m}\mathrm{H}.$$

2.3. **Example** (Deligne, Steenbrink). Suppose X a smooth and proper variety, and suppose $D \subset X$ a divisor with normal crossings. Then the integral cohomology

$$\mathrm{H}^k(X - D, \mathbf{Z})$$

admits a canonical mixed Hodge structure.

Sketch of proof. Denote by $j\colon X-D \hookrightarrow X$ the open immersion. The *logarithmic de Rham complex* is the subcomplex $\Omega_X\langle\log D\rangle \subset j_\star\Omega_{X-D}$ consisting of those ω such that both ω and $d\omega$ have at most a pole of order one on D. The inclusion $\Omega_X\langle\log D\rangle \hookrightarrow j_\star\Omega_{X-D}$ is a quasi-isomorphism, whence:

$$\mathrm{H}^k(X-D,\mathbf{C}) \cong \mathrm{H}^k(X-D,\Omega_{X-D}) \cong \mathrm{H}^k(X,j_\star\Omega_{X-D}) \cong \mathrm{H}^k(X,\Omega_X\langle\log D\rangle).$$

Again the “foolish” filtration on $\Omega_X\langle\log D\rangle$ gives rise to a degenerating spectral sequence on the cohomology, yielding a decomposition

$$\mathrm{H}^k(X,\Omega_X\langle\log D\rangle) = \bigoplus_{p+q=k}\mathrm{H}^q(X,\Omega^p_X\langle\log D\rangle).$$

It remains to construct the weight filtration on $\mathrm{H}^k(X-D,\mathbf{Z})$ with the desired properties. The plan now is to construct a complex E of sheaves of abelian groups on X along with a map $E \longrightarrow Rj_\star\mathbf{Z}_{X-D}$ such that:

* the cohomology sheaves $\mathscr{H}^m_X(E)$ will be 0 unless $0 \leq m \leq n$, in which case the morphism $\mathscr{H}^m(E) \longrightarrow R^m j_\star\mathbf{Z}_{X-D}$ will be an isomorphism; and
* there is a quasi-isomorphism $\Lambda(e)\otimes\mathbf{C} \xrightarrow{\sim} \Omega_X\langle\log D\rangle$.

The weight filtration will then be induced by the canonical filtration of E.

Let $\mathscr{M}_{X,D} := \mathscr{O}_X \cap j_\star\mathscr{O}^\times_{X-D}$ be the log structure on X generated by D. Form the exponential $e\colon \mathscr{O}_X \longrightarrow \mathscr{M}^{\mathrm{gp}}_{X,D}$ given by

$$e(g) := \exp(2\pi i g).$$

Regard e as a two term complex, quasi-isomorphic to $\tau_{\leq 1}Rj_\star\mathbf{Z}_{X-D}$. The Koszul complex $\Lambda(e)$ of e is given by

$$\Lambda(e)^p := \Gamma_{n-p}\mathscr{O}_X \otimes_{\mathbf{Z}_X} \Lambda^p\mathscr{M}^{\mathrm{gp}}_{X,D},$$

where $n = \dim_{\mathbf{C}} X$. [Properly speaking, this cannot be done until e is replaced with a complex in which both terms are torsion-free.] One obtains a map

$$\Lambda(e) \longrightarrow Rj_\star\mathbf{Z}_{X-D}$$

with our desired properties.

Let us construct the map $\varphi\colon \Lambda(e) \longrightarrow \Omega_X\langle\log D\rangle$ explicitly:

$$\varphi(x^{[n-p]}\otimes(y_1\wedge\cdots\wedge y_p)) = (2\pi i)^{-p}\frac{x^{n-p}}{(n-p)!}d\log y_1\wedge\cdots\wedge d\log y_p,$$

which actually induces a quasiisomorphism $\Lambda(e)\otimes\mathbf{C} \xrightarrow{\sim} \Omega_X\langle\log D\rangle$. □

Let us note that under this quasi-isomorphism, the canonical filtration on $\Lambda(e)\otimes\mathbf{C}$ corresponds to a filtration $W_*\Omega_X\langle\log D\rangle$ in which:

$$W_m\Omega^p_X\langle\log D\rangle = \begin{cases} 0 & \text{if } m<0;\\ \Omega^{p-m}_X\wedge\Omega^m_X\langle\log D\rangle & \text{if } 0\leq m\leq p;\\ \Omega^p_X\langle\log D\rangle & \text{if } m>p.\end{cases}$$

These Hodge structures can also be found on the cohomology of simplicial varieties. We mention an example of interest.

2.4. **Example.** Suppose G a complex linear algebraic group. Then any even cohomology group $\mathrm{H}^{2k}(B_*G, \mathbf{Z})$ is pure of type (n,n). One can prove this by reducing first to the case tori and then to $\mathbf{G}_m$ itself. The only potentially nonzero Hodge numbers of $H^1(\mathbf{G}_m, \mathbf{Z}) = \mathbf{Z}$ are h^{11} and $h^{01} = h^{10}$, whence one deduces that $h^{11} = 1$.

If we consider in particular the case $G = \mathbf{GL}_n$, then we find that the universal Chern classes $c_k \in \mathrm{H}^{2k}(B_*\mathbf{GL}_N, \mathbf{Q})$ are of type (k,k). Consequently, one may deduce for any smooth complex algebraic variety X, the image of the Chern character

$$\chi_i \colon K_i(X) \otimes \mathbf{Q} \longrightarrow \bigoplus_{j \in \mathbf{Z}} \mathrm{H}^{2j-i}(X, \mathbf{Q})$$

must lie in

$$\bigoplus_{j \in \mathbf{Z}} (W_{2j}\mathrm{H}^{2j-i}(X, \mathbf{Q}) \cap F^i\mathrm{H}^{2j-i}(X, \mathbf{C})).$$

If X is projective, then in fact $\chi_i = 0$ for $i > 0$. Deligne cohomology actually makes use of these Hodge conditions to extract finer invariants of K-theory and thus of cycle classes.

3. Cycle classes and the Abel–Jacobi map

3.1. Suppose X a smooth projective variety over $\mathbf{C}$ of dimension n. The *cycle class map*

$$z^m_{\mathrm{cyc}} \colon \mathbf{CH}^m(X) \longrightarrow \mathrm{H}^{2m}(X, \mathbf{Z}(m))$$

is constructed in the following manner. Suppose $Z \subset X$ a smooth codimension m subvariety. Then we have the Thom isomorphism

$$\vartheta \colon \mathrm{H}^{2m}(X, X - Z; \mathbf{Z}(m)) \xrightarrow{\sim} \mathrm{H}^0(Z; \mathbf{Z}) \cong \mathbf{Z}.$$

We also have the restriction map

$$\rho \colon \mathrm{H}^{2m}(X, X - Z; \mathbf{Z}(m)) \longrightarrow \mathrm{H}^{2m}(X; \mathbf{Z}(m)).$$

Now we let $z^m_{\mathrm{cyc}}(Z) = \rho\vartheta^{-1}(1)$.

We can also find an m-form ω that will be de Rham representative of $z^m_{\mathrm{cyc}}(Z)$ in the following manner. Let V be a neighborhood of the zero section of the normal bundle $N_{Z/X}$ that can be identified with the corresponding neighborhood of Z. Now let ω be any *real* m-form whose support is in V and is compact over Z, such that for any $z \in Z$,

$$\int_{V_z} \omega = 1.$$

Extend ω from U to all of X by 0. The image of ω in $\mathrm{H}^m(X, \mathbf{R})$ represents the image of $z^m_{\mathrm{cyc}}(Z)$ in $\mathrm{H}^m(X, \mathbf{R})$.

Now we can work out the position of $z^m_{\mathrm{cyc}}(Z)$ is the Hodge filtration by Poincaré duality. Indeed, one has

$$\int_X \omega \wedge \beta = \int_U \omega \wedge \beta = \int_Z i^\star \beta$$

for any $\beta \in \mathrm{H}^{2n-2m}(X, \mathbf{C})$. Since U retracts onto Z, there exists an $(m-1)$-form μ on U such that

$$\beta|_U = \pi^\star(\beta|_Z) + d\mu;$$

hence

$$\int_X \omega \wedge \beta = \int_U \omega \wedge \pi^\star(\beta|_Z).$$

Now since $\int_{V_z} \omega = 1$, we get

$$\int_X \omega \wedge \beta = \int_Z \beta|_Z.$$

This integral vanishes unless β is of type $(n-m, n-m)$; hence $z^m_{\mathrm{cyc}}(Z)$ is of type (m, m). Consequently, we have defined a map

$$z^m_{\mathrm{cyc}} \colon \mathbf{CH}^m(X) \longrightarrow \mathbf{Hdg}^m(X),$$

where

$$\begin{aligned} \mathbf{Hdg}^m(X) &:= \mathrm{H}^{2m}(X, \mathbf{Z}(m)) \cap \mathrm{H}^m(X, \Omega^m) \\ &\cong \ker \left[\mathrm{H}^{2m}(X, \mathbf{Z}(m)) \longrightarrow \frac{\mathrm{H}^{2m}(X, \mathbf{C})}{F^m \mathrm{H}^{2m}(X, \mathbf{C})} \right]. \end{aligned}$$

3.2. **Conjecture** (Hodge Conjecture). *For any smooth projective X, the rational cycle class map*

$$z^m_{\mathrm{cyc}} \otimes \mathbf{Q} \colon \mathbf{CH}^m(X) \otimes \mathbf{Q} \longrightarrow \mathbf{Hdg}^m(X) \otimes \mathbf{Q}$$

is surjective.

3.3. **Theorem** (Lefschetz, Hodge). *The cycle class map*

$$z^1_{\mathrm{cyc}} \colon \mathbf{CH}^1(X) \cong \mathbf{Pic}(X) \longrightarrow \mathbf{Hdg}^1(X)$$

is surjective, and its kernel is $\mathbf{Pic}^0(X)$. *Consequently,* $\mathbf{Hdg}^1(X) \cong \mathbf{NS}(X)$.

We can also ask about the kernel of the map in general. This is where intermediate Jacobians make their appearance.

3.4. **Definition.** Suppose H a torsion-free mixed Hodge structure. For any integer m, the m-th *Jacobian* of H is

$$J^m(\mathrm{H}) := \mathrm{H}_{\mathbf{C}}/(F^m\mathrm{H}_{\mathbf{C}} + \mathrm{H}_{\mathbf{Z}}).$$

Note that

$$J^m(\mathrm{H}) \cong J^0\underline{\mathrm{Hom}}(\mathbf{Z}, \mathrm{H}(m)) \cong J^0\underline{\mathrm{Hom}}(\mathbf{Z}(-m), \mathrm{H}).$$

3.5. **Proposition.** *If* H *is a mixed Hodge structure such that* $W_k\mathrm{H} = \mathrm{H}$ *(we say* H *is "of highest weight* k*"), then for any* $m > k/2$*, the Jacobian* $J^m\mathrm{H}$ *is a generalized complex torus.*

Proof. We first claim that the map $\mathrm{H}_{\mathbf{R}} := \mathrm{H}_{\mathbf{Z}} \otimes \mathbf{R} \longrightarrow \mathrm{H}_{\mathbf{C}}/F^m\mathrm{H}$ is an injection. Indeed, an element of the kernel lies in $F^m\mathrm{H} \cap \mathrm{H}_{\mathbf{R}}$ and thus also in $F^m\mathrm{H} \cap \overline{F^m\mathrm{H}}$. But the latter is 0. (For this, note if $x \in F^m\mathrm{H} \cap \overline{F^m\mathrm{H}}$ is of weight h, then its projection to gr^W_h must be a sum of components of type (a, b), where $a \geq m$ and $b \geq m$, whence $h \geq 2m > k$.)

Now let K be a complement of $\mathrm{H}_{\mathbf{R}}$ in the $\mathbf{R}$-vector space $\mathrm{H}_{\mathbf{C}}/F^m\mathrm{H}$. Then we get

$$J^m\mathrm{H} \cong (\mathrm{H}_{\mathbf{R}}/\mathrm{H}_{\mathbf{Z}}) \oplus K \cong (S^1)^r \times \mathbf{R}^s. \qquad \square$$

If $\mathrm{H}_{\mathbf{C}} = F^m\mathrm{H} \oplus \overline{F^m\mathrm{H}}$, then $s = 0$ in the proof above. Consequently, we have:

3.5.1. **Corollary.** *If* H *is a pure Hodge structure of weight* $2m-1$*, then* $J^m(\mathrm{H})$ *is a complex torus.*

It turns out that these Jacobians are a handy way of computing the Ext groups of mixed Hodge structures.

3.6. **Theorem** (Carlson). *Suppose* H' *and* H'' *two torsion-free mixed Hodge structures. Then there is a canonical isomorphism*

$$\mathrm{Ext}^1(\mathrm{H}', \mathrm{H}'') \cong \frac{W_0\underline{\mathrm{Hom}}(\mathrm{H}', \mathrm{H}'')}{(W_0\underline{\mathrm{Hom}}(\mathrm{H}', \mathrm{H}'')_{\mathbf{C}} \cap F^0\underline{\mathrm{Hom}}(\mathrm{H}', \mathrm{H}'')) + W_0\underline{\mathrm{Hom}}(\mathrm{H}', \mathrm{H}'')_{\mathbf{Z}}}.$$

In particular, we have:

3.6.1. **Corollary.** *If for some* m *we have* $W_m\mathrm{H}' = 0$ *and* $W_m\mathrm{H}'' = \mathrm{H}''$ *(that is, if the weight of* H' *are all greater than the weights of* H''*), then we have a canonical isomorphism*

$$\mathrm{Ext}^1(\mathrm{H}', \mathrm{H}'') \cong J^0\underline{\mathrm{Hom}}(\mathrm{H}', \mathrm{H}''),$$

and this is a generalized complex torus.

3.6.2. **Corollary.** *For any mixed Hodge structure* H*, the functor* $\mathrm{Ext}^1(\mathrm{H}, -)$ *is right exact. In particular,* $\mathrm{Ext}^r(\mathrm{H}, -) = 0$ *for* $r \geq 2$.

3.7. **Example.** If $m < n$, then

$$\mathrm{Ext}^1(\mathbf{Z}(m), \mathbf{Z}(n)) \cong \mathbf{C}/(2\pi\sqrt{-1})^{n-m}\mathbf{Z}.$$

3.8. **Example.** If H is a pure Hodge structure of weight $2m-1$, then

$$J^m(\mathrm{H}) \cong J^0\underline{\mathrm{Hom}}(\mathbf{Z}(-m),\mathrm{H}) \cong \mathrm{Ext}^1(\mathbf{Z}(-m),\mathrm{H}).$$

3.9. **Definition.** Suppose X a smooth projective variety over $\mathbf{C}$. Then

$$J^m(X) := J^m(\mathrm{H}^{2m-1}(X)) \cong \frac{\mathrm{H}^{2m-1}(X,\mathbf{C})}{F^m\mathrm{H}(X,\mathbf{C}) \oplus \mathrm{H}^{2m-1}(X,\mathbf{Z})}$$

is called the *intermediate Jacobian* of Griffiths.

When $m = 1$, we obtain the *Picard variety* $J^1(X) = \mathbf{Pic}^0(X)$; when $m = \dim_{\mathbf{C}} X$, we obtain the *Albanese variety* $J^m(X)$ of X.

If $n = \dim_{\mathbf{C}} X$, Poincaré duality gives an alternate description:

$$J^m(X) \cong \frac{F^{n-m+1}\mathrm{H}^{2n-2m+1}(X)^\vee}{H_{2n-2m+1}(X,\mathbf{Z})}.$$

3.10. Suppose again X a smooth projective variety over $\mathbf{C}$ of dimension n. The kernel

$$\mathbf{CH}^m_0(X) := \ker\left[z^m_{\mathrm{cyc}} \colon \mathbf{CH}^m(X) \longrightarrow \mathbf{Hdg}^m(X)\right]$$

consists of cycles Z such that $Z = \partial W$ for some W. Integration along this W yields an element

$$\int_W \in \Omega^{2n-2m+1}(X)^\vee$$

On an exact form $d\psi$ with $\psi \in \Omega^{2n-2m}$, we have

$$\int_W d\psi = \int_Z \psi,$$

which vanishes unless if $\psi \in F^{n-m+1}\Omega^{2n-2m}_X$. Consequently, $\int_W$ defines an element of $F^{n-m+1}\mathrm{H}(X,\mathbf{C})^\vee$. Now if we chose a different W' such that $Z = \partial W'$, then $W - W'$ is a class of $H_{2n-2m+1}(X,\mathbf{Z})$, whence we get a well-defined class

$$\varphi^m_X(Z) \in \frac{F^{n-m+1}\mathrm{H}^{2n-2m+1}(X)^\vee}{H_{2n-2m+1}(X,\mathbf{Z})} = J^m(X).$$

Furthermore, if Z and Z' are rationally equivalent, then $\varphi^m_X(Z) = \varphi^m_X(Z')$. (In effect, if Z and Z' are connected by a family of cycles parametrized by $\mathbf{P}^1$, we use this perscription to obtain a *holomorphic* map $\mathbf{P}^1 \longrightarrow J^m(X)$, which must be constant.)

We have just defined a map called the *Abel–Jacobi map*

$$\varphi^m_X \colon \mathbf{CH}^m_0(X) \longrightarrow J^m(X).$$

3.11. **Example** (Bloch–Srinivas, Murre)**.** Suppose X a smooth complete intersection of dimension n with $h^{0n} = 0$. Then the Abel–Jacobi map

$$\varphi^2_X \colon \mathbf{CH}^2_0(X) \longrightarrow J^2(X)$$

is an isomorphism.

4. Deligne cohomology

Deligne cohomology can be thought of as a systematic way of packaging both ordinary cohomology and the intermediate Jacobians.

4.1. **Definition.** For any integer p, the *Deligne complex* $\mathbf{Z}(p)_{\mathrm{D}}$ is a sheaf of complexes (or, better, simplicial abelian groups, or, still better, spectra) on the site of complex analytic manifolds defined as the homotopy fiber product:

$$\mathbf{Z}(p)_{\mathrm{D}} := \mathbf{Z}(p) \times^h_{\mathbf{C}} \Omega^{\geq p}.$$

Its cohomology groups on a compact complex analytic manifold X are the *Deligne cohomology groups*:

$$\mathrm{H}^q_{\mathrm{D}}(X, \mathbf{Z}(p)) := \mathrm{H}^q(X, \mathbf{Z}(p)_{\mathrm{D}}).$$

4.2. Equivalently, $\mathbf{Z}(p)_{\mathrm{D}}$ is the fiber of the natural map $\mathbf{Z}(p) \longrightarrow \mathbf{C}/F^p\Omega$.

One can form an explicit complex of sheaves of abelian groups on a complex analytic manifold that represents $\mathbf{Z}(p)_{\mathrm{D}}$:

$$0 \longrightarrow \mathbf{Z}_X(p) \longrightarrow \mathscr{O}_X \longrightarrow \Omega^1_X \longrightarrow \cdots \longrightarrow \Omega^{p-1}_X \longrightarrow 0.$$

4.3. **Example.** Of course $\mathbf{Z}(0)_{\mathrm{D}} = \mathbf{Z}$, and so

$$\mathrm{H}^q_{\mathrm{D}}(X, \mathbf{Z}(0)) \cong \mathrm{H}^q(X, \mathbf{Z}).$$

4.4. **Example.** The map

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{Z}(1) & \longrightarrow & \mathscr{O}_X & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} & & \downarrow \\ 1 & \longrightarrow & 1 & \longrightarrow & \mathscr{O}_X^\times & \longrightarrow & 1 \end{array}$$

is an equivalence $\exp\colon \mathbf{Z}_X(1)_{\mathrm{D}} \xrightarrow{\sim} \mathscr{O}_X^\times[-1]$, and so

$$\mathrm{H}^q_{\mathrm{D}}(X, \mathbf{Z}(1)) \cong \mathrm{H}^{q-1}(X, \mathscr{O}_X^\times).$$

In particular, note that $\mathrm{H}^2_{\mathrm{D}}(X, \mathbf{Z}(1)) \cong \mathbf{Pic}(X)$, and we have an exact sequence

$$0 \longrightarrow J^1(X) \longrightarrow \mathrm{H}^2_{\mathrm{D}}(X, \mathbf{Z}(1)) \longrightarrow \mathbf{NS}(X) \longrightarrow 0.$$

4.5. **Example.** There is an equivalence

$$\mathbf{Z}_X(2)_{\mathrm{D}} \simeq \left[d\log\colon \mathscr{O}_X^\times \longrightarrow \Omega^1_X\right][-1].$$

One can show that $\mathrm{H}^2_{\mathrm{D}}(X, \mathbf{Z}(2))$ is the group of line bundles with holomorphic connection.

4.6. The long exact sequence of the fiber gives

$$\mathrm{H}^{q-1}(X, \mathbf{Z}(p)) \longrightarrow \frac{\mathrm{H}^{q-1}(X, \mathbf{C})}{F^p\mathrm{H}^{q-1}(X, \mathbf{C})} \longrightarrow \mathrm{H}^q_{\mathrm{D}}(X, \mathbf{Z}(p)) \longrightarrow \mathrm{H}^q(X, \mathbf{Z}(p)) \longrightarrow \frac{\mathrm{H}^q(X, \mathbf{C})}{F^p\mathrm{H}^q(X, \mathbf{C})}.$$

When $q < 2p$, the map on the right is an injection, so we have a short exact sequence

$$0 \longrightarrow J^p \mathrm{H}^{q-1}(X, \mathbf{Z}(p)) \longrightarrow \mathrm{H}^q_{\mathrm{D}}(X, \mathbf{Z}(p)) \longrightarrow \mathrm{H}^q(X, \mathbf{Z}(p)) \longrightarrow 0.$$

When $q = 2p$, we get a short exact sequence

$$0 \longrightarrow J^p(X) \longrightarrow \mathrm{H}^{2p}_{\mathrm{D}}(X, \mathbf{Z}(p)) \longrightarrow \mathbf{Hdg}^p(X) \longrightarrow 0.$$

Deligne–Beilinson cohomology is the extension of Deligne cohomology to smooth quasiprojective varieties. The strategy is exactly the same as the strategy to construct the mixed Hodge structure on the singular cohomology of a smooth quasiprojective varieties. We discuss the foundations only briefly.

4.7. **Definition.** By a *good compactification*

$$(U, X) = [\, j \colon U \subset X \,],$$

we will mean a smooth, proper variety X with a Zariski open $j \colon U \subset X$ such that $D = X - U$ is a divisor with normal crossings.

For any integer p, the *Deligne–Beilinson complex* $\mathbf{Z}(p)_{\mathrm{D}}$ is the following sheaf of complexes (or simplicial abelian groups or spectra) on the site of good compactifications:

$$\mathbf{Z}(p)_{\mathrm{DБ}}(U, X) := Rj_\star \mathbf{Z}(p) \times^h_{Rj_\star \mathrm{C}} \Omega^{\geq p}_X \langle \log D \rangle.$$

Its cohomology groups on a given compactification are the *Deligne–Beilinson cohomology groups*:

$$\mathrm{H}^q_{\mathrm{DБ}}((U, X), \mathbf{Z}(p)) := \mathrm{H}^q(X, \mathbf{Z}(p)_{\mathrm{DБ}}).$$

4.8. For proper maps $g \colon Y \longrightarrow X$ of relative dimension d, there are *transfer maps*

$$g_\star \colon \mathbf{Z}(p)_Y \times^h_{\mathrm{C}_Y} \Omega^{\geq p}_Y = \mathbf{Z}(p)_{\mathrm{D},Y} \longrightarrow \mathbf{Z}(p-d)_{\mathrm{D},X}[-2d] = \mathbf{Z}(p-d)_X \times^h_{\mathrm{C}_X} \Omega^{\geq p-d}_X[-2d];$$

these are just given by "integration along the fiber." In the bundle case, this is easy to write:

$$[Y_+, \mathbf{Z}(p)_{\mathrm{D}}] \simeq [\mathrm{Th}(g)_+, \Sigma^{N-2d}\mathbf{Z}(p-d)_{\mathrm{D}}] \longrightarrow [\Sigma^N X_+, \Sigma^{N-2d}\mathbf{Z}(p-d)_{\mathrm{D}}] \simeq [X_+, \Sigma^{-2d}\mathbf{Z}(p-d)_{\mathrm{D}}]$$

4.9. Suppose X a smooth projective variety. We have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{CH}^m_0(X) & \longrightarrow & \mathbf{CH}^m(X) & \xrightarrow{z^m_{\mathrm{cyc}}} & \mathbf{Hdg}^m(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi^m_X} & & & & \| & & \\ 0 & \longrightarrow & J^m(X) & \longrightarrow & \mathrm{H}^{2m}_{\mathrm{D}}(X, \mathbf{Z}(m)) & \longrightarrow & \mathbf{Hdg}^m(X) & \longrightarrow & 0 \end{array}$$

4.10. **Theorem.** *Suppose X a smooth projective variety. There exists a cycle class map*

$$\gamma^m_{\mathrm{cyc}} \colon \mathbf{CH}^m(X) \longrightarrow \mathrm{H}^{2m}_{\mathrm{D}}(X, \mathbf{Z}(m))$$

completing this diagram.

Construction. For any smooth subvariety $i\colon Z \subset X$ of codimension m, define $\gamma^m_{\mathrm{cyc}}(Z)$ as the image of 1 under

$$i_\star\mathbf{Z} \cong \mathrm{H}^0_{\mathrm{D}}(Z, \mathbf{Z}) \longrightarrow \mathrm{H}^{2m}_{\mathrm{D}}(X, \mathbf{Z}(m)).$$

□