THE ATIYAH–HIRZEBRUCH SPECTRAL SEQUENCE FOR ALGEBRAIC K-THEORY

by

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These are notes for a talk for ΘΣ. I'll describe a weight filtration on the algebraic K-theory of a regular scheme, due to Grayson. I'll describe it again using the slice filtration of Voevodsky. Finally, I'll sketch a proof that the graded pieces of this filtration are given by motivic cohomology, in the sense described in Jacob Lurie's lecture.

Contents

1. Motivation from topology

Notation 1.1. — For any spectrum E and any space (i.e., simplicial set) X, write $E(X)$ for the function spectrum $\mathbf{F}(\Sigma^\infty X_+, E)$, and write

$$E^*(X) = E_{-*}(X) = \pi_{-*}E(X).$$

1.2. — The skeletal filtration

$$X^0 \subset X^1 \subset \cdots \subset X^{n-1} \subset X^n \subset \cdots \subset X$$

induces a limit sequence

$$E(X) \longrightarrow \cdots \longrightarrow E(X^n) \longrightarrow E(X^{n-1}) \longrightarrow \cdots \longrightarrow E(X^1) \longrightarrow E(X^0),$$

whence, if $\lim^1_{r\geq 1} E_r^{s,t} = 0$, we have a strongly convergent spectral sequence

$$E_1^{s,t} = E^{s+t}(X^s/X^{s-1}) \Longrightarrow E^{s+t}(X),$$

called the *Atiyah–Hirzebruch spectral sequence*.

Lemma 1.3. — *One may identify the E_1 term thus:*

$$E_1^{s,t} = E^{s+t}(X^s/X^{s-1}) \cong \widetilde{H}^s(X^s/X^{s-1}, E^t).$$

Lemma 1.4. — *For any abelian group π, the cohomology of the complex*

$$\cdots \longrightarrow \widetilde{H}^s(X^s/X^{s-1},\pi) \longrightarrow \widetilde{H}^{s+1}(X^{s+1}/X^s,\pi) \longrightarrow \cdots,$$

where the differential is the composite

$$\widetilde{H}^s(X^s/X^{s-1},\pi) \longrightarrow H^{s+1}(X^{s+1},\pi) \longrightarrow \widetilde{H}^{s+1}(X^{s+1}/X^s,\pi),$$

is precisely $H^*(X,\pi)$.

Corollary 1.5. — *The E_2 page of the Atiyah–Hirzebruch spectral sequence can be identified thus:*

$$E_2^{s,t} \cong H^s(X, E^t) \Longrightarrow E^{s+t}(X).$$

Example 1.6. — When E is even periodic, this spectral sequence is particularly simple. In particular, for complex K-theory, one has

$$E_2^{s,t} \cong \begin{Bmatrix} H^s(X,\mathbf{Z}) & \text{if } t \text{ is even;} \\ 0 & \text{if } t \text{ is odd.} \end{Bmatrix} \Longrightarrow KU^{s+t}(X).$$

The differentials of this spectral sequence are torsion; hence it degenerates rationally.

1.7. — Inspired by this observation, Beilinson offered a provisional definition of motivic cohomology with rational coefficints as the weight j Adams eigenspace

$$H^i(X, \mathbf{Q}(j)) = K_{2j-i}(X)^{(j)}_{\mathbf{Q}}.$$

2. K-theory as a $(1,1)$-periodic $\mathbf{P}^1$-spectrum

Suppose S a separated, noetherian scheme of finite Krull dimension. Then $K\colon X \longmapsto K(X)$ defines a presheaf of spectra on the category (Sch/S) of noetherian schemes of finite Krull dimension over S.

***Theorem 2.1* (Nisnevich descent).** — *The presheaf K satisfies Nisnevich descent on* (Sch/S).

Corollary 2.2. — *The presheaf K extends uniquely to a functor*

$$K\colon \mathscr{S}(\mathrm{Sm}/S)^{\mathrm{op}}_{\mathrm{Nis}} \longrightarrow \mathscr{S}\!p$$

that sends colimits of sheaves on the Nisnevich site $(\mathrm{Sm}/S)_{\mathrm{Nis}}$ *of smooth, noetherian S-schemes of finite Krull dimension to limits of spectra.*

***Proposition 2.3* (Homotopy invariance).** — *On regular schemes, algebraic K-theory is $\mathbf{A}^1$-invariant; that is, for any regular scheme X, the projection $X \times \mathbf{A}^1 \longrightarrow X$ induces an equivalence $K(X) \simeq K(X \times \mathbf{A}^1)$.*

Corollary 2.4. — *The presheaf K descends uniquely to a functor*

$$K\colon L_{\mathbf{A}^1}\mathscr{S}(\mathrm{Sm}/S)^{\mathrm{op}}_{\mathrm{Nis}} \longrightarrow \mathscr{S}\!p$$

that sends colimits in $L_{\mathbf{A}^1}\mathscr{S}(\mathrm{Sm}/S)_{\mathrm{Nis}}$ to limits of spectra.

Corollary 2.5. — *The presheaf K extends to a unique pointed functor*

$$\widetilde{K}\colon (\star/L_{\mathbf{A}^1}\mathscr{S}(\mathrm{Sm}/S)_{\mathrm{Nis}})^{\mathrm{op}} \longrightarrow \mathscr{S}p$$

that sends colimits to limits such that for any smooth S-scheme X, one has $\widetilde{K}(X_+) = K(X)$.

Corollary 2.6. — *The functor $\Omega^\infty\mathscr{K}$ is representable; that is, there is a unique $\mathbf{A}^1$-invariant sheaf* BGL *and an equivalence of Nisnevich sheaves*

$$\Omega^\infty\mathscr{K} \simeq \mathrm{Map}_{(\star/L_{\mathbf{A}^1}\mathscr{S}(\mathrm{Sm}/S)_{\mathrm{Nis}})}\left((-)_+, \mathrm{BGL}\right).$$

2.7. — To construct BGL, one may begin by contemplating the sheaf $B\,\mathrm{GL}_* = \coprod_{n\geq 0} B\,\mathrm{GL}_n$. This is an E_∞ monoid in $L_{\mathbf{A}^1}\mathscr{S}(\mathrm{Sm}/S)_{\mathrm{Nis}}$. Hence it admits a classifying space $B(B\,\mathrm{GL}_*)$ and a group completion $\Omega B(B\,\mathrm{GL}_*)$. One sees, almost by definition, that

$$\mathrm{BGL} \simeq \Omega B(B\,\mathrm{GL}_*).$$

It is also not difficult to construct an equivalence

$$B\,\mathrm{GL}_\infty \times \mathbf{Z} \simeq \Omega B(B\,\mathrm{GL}_*).$$

Here, the main point is that each $B\,\mathrm{GL}_n$ is $\mathbf{A}^1$-connected; this follows from the fact that for any Nisnevich sheaf X, the morphism $\widetilde{\pi}_0(X) \longrightarrow \widetilde{\pi}_0^{\mathbf{A}^1}(X)$ of sheaves of sets is an epimorphism.

Consider the Grassmannian of k-planes in N-space $G_S(k,N)$. One can form the colimits

$$G_S(k,\infty) = \mathrm{colim}_{N\geq k}\, G_S(k,N)$$

as well as

$$G_S(\infty,\infty) = \mathrm{colim}_{k\geq 0}\, G_S(k,\infty) = \mathrm{colim}_{N\geq k\geq 0}\, G_S(k,N)$$

as ind-schemes. It is not hard to see that $G_S(k,N)$ is the quotient $(U_{k,N}/\,\mathrm{GL}_k)_{\text{ét}}$, where $U_{k,N}$ is the scheme of monomorphisms $\mathscr{O}_S^k \hookrightarrow \mathscr{O}_S^N$. Likewise $G_S(k,\infty)$ is the quotient $(U_{k,\infty}/\,\mathrm{GL}_k)_{\text{ét}}$, and this quotient is in turn a model for $p_\star p^\star B\,\mathrm{GL}_k$, where p is the projection $(\mathrm{Sm}/S)_{\text{ét}} \longrightarrow (\mathrm{Sm}/S)_{\mathrm{Nis}}$. By Hilbert Theorem 90, we now have

$$G_S(k,\infty) \simeq (U_{k,\infty}/\,\mathrm{GL}_k)_{\text{ét}} \simeq p_\star p^\star B\,\mathrm{GL}_k \simeq B\,\mathrm{GL}_k$$

We conclude that $G_S(\infty,\infty)\times\mathbf{Z}$ represents the K-theory space functor in the sense that there is a equivalence of Nisnevich sheaves

$$\Omega^\infty\mathscr{K} \simeq \mathrm{Map}_{(\star/L_{\mathbf{A}^1}\mathscr{S}(\mathrm{Sm}/S)_{\mathrm{Nis}})}((-)_+, G_S(\infty,\infty)\times\mathbf{Z})$$

***Proposition 2.8* (Projective bundle).** — *Suppose V a vector bundle of rank $r+1$ on a noetherian scheme X of finite Krull dimension. Then there is a canonical equivalence*

$$K(\mathbf{P}_X V) \simeq K(X)^{\vee(r+1)}.$$

In particular, $K(\mathbf{P}^1\times X) \simeq K(X)\vee K(X)$.

Corollary 2.9. — *In particular, for any pointed smooth scheme (X,x), one has*

$$\widetilde{K}\left(\mathbf{P}^1\wedge(X,x)\right) \simeq \widetilde{K}(X,x).$$

(Here we think of $\mathbf{P}^1$ as pointed at ∞.)

Corollary 2.10. — *The functor $\widetilde{K}$ extends canonically to a unique stable functor*

$$\widetilde{K}\colon \mathscr{S}p_{\mathbf{P}^1}(\star/L_{\mathbf{A}^1}\mathscr{S}(\mathrm{Sm}/S)_{\mathrm{Nis}})^{\mathrm{op}} \longrightarrow \mathscr{S}p$$

that sends colimits to limits such that for any pointed smooth S-scheme (X,x), one has $\widetilde{K}(\Sigma^{\infty}_{\mathbf{P}^1}(X,x)) = \widetilde{K}(X,x)$.

Corollary 2.11. — *There exists a $\mathbf{P}^1$-spectrum $\mathbf{BGL} \in \mathscr{S}p_{\mathbf{P}^1}(\star/L_{\mathbf{A}^1}\mathscr{S}(\mathrm{Sm}/S)_{\mathrm{Nis}})$ such that for any smooth S-scheme X,*

$$K^{p-q}(X) = [\Sigma^{\infty}_{\mathbf{P}^1}X_+, S^p \wedge \mathbf{G}_m^{\wedge q} \wedge \mathbf{BGL}].$$

Moreover, $\mathbf{BGL}$ *is* $(1,1)$*-periodic in the sense that there is a canonical equivalence*

$$\mathbf{BGL} \simeq \mathbf{BGL} \wedge \mathbf{P}^1 \simeq \mathbf{BGL} \wedge S^1 \wedge \mathbf{G}_m.$$

2.12. — We way construct $\mathbf{BGL}$ using BGL in the following manner. Observe that

$$\begin{aligned}
\mathrm{Map}(\mathbf{P}^1 \wedge \mathrm{BGL}, \mathrm{BGL}) &\simeq \lim_{N \geq k \geq 0} \mathrm{Map}(\mathbf{P}^1 \wedge G_S(k,N), \mathrm{BGL}) \\
&\simeq \lim_{N \geq k \geq 0} \widetilde{K}(\mathbf{P}^1 \wedge G_S(k,N)) \\
&\simeq \lim_{N \geq k \geq 0} \widetilde{K}(G_S(k,N)) \\
&\simeq \lim_{N \geq k \geq 0} \widetilde{K}(G_S(k,N)) \\
&\simeq \lim_{N \geq k \geq 0} \mathrm{Map}(G_S(k,N), \mathrm{BGL}) \\
&\simeq \mathrm{Map}(\mathrm{BGL}, \mathrm{BGL}).
\end{aligned}$$

Now we may contemplate the map $\alpha\colon \mathbf{P}^1 \wedge \mathrm{BGL} \longrightarrow \mathrm{BGL}$ that corresponds to the identity under the identifications above. Now it is easy to check that $\mathbf{BGL}$ is the "constant" $\mathbf{P}^1$-spectrum whose structrue maps are all α.

3. Grayson's filtration by commuting automorphisms

Suppose X a quasiseparated, quasicompact scheme. Goodwillie and Lichtenbaum introduced a exhaustive filtration on the homotopy K-theory of X:

$$\cdots \longrightarrow W^2KH(X) \longrightarrow W^1KH(X) \longrightarrow W^0KH(X) = KH(X).$$

3.1. — For any two quasicompact and quasiseparated schemes X and Y, define the ∞-category $\mathscr{P}(X,Y)$ as the ∞-category of pseudocoherent complexes M on $X \times Y$ such that $\operatorname{supp} M$ is finite over X and $\mathrm{pr}_{1,\star}M$ is a perfect complex on X. We contemplate the *bivariant K-theory spectrum*

$$K(X,Y) := K\mathscr{P}(X,Y).$$

Note that $K(X, \operatorname{Spec}\mathbf{Z}) = K(X)$ and $K(\operatorname{Spec}\mathbf{Z}, Y) = G(Y)$. Observe also that the assignment $(M,N) \longmapsto \mathrm{pr}_{13,\star}(\mathrm{pr}_{12}^{\star}M \otimes \mathrm{pr}_{23}^{\star}N)$ defines a morphism $K(X,Y) \wedge K(Y,Z) \longrightarrow K(X,Z)$. One can show that this gives the category of quasicompact and quasiseparated schemes the structure of a category enriched in spectra.

Now for a fixed quasicompact and quasiseparated scheme X, define, for any finite set I, the *dual I-th cross-effects* $\mathrm{cr}^I K(X;-)\colon (\star/\mathrm{Sch})^{\times I} \longrightarrow \mathcal{S}p$ as the functor

$$\mathrm{cr}^I K(X;Y_I) := \mathrm{cofib}\left[\operatorname*{colim}_{J\subsetneq I} K\left(X, \prod_{j\in J} Y_j\right) \longrightarrow K\left(X, \prod_{i\in I} Y_i\right)\right].$$

Now write

$$W^I KH(X) := \operatorname*{colim}_{\Delta} \mathrm{cr}^I K(\Delta^\bullet_X; \mathbf{P}^1, \mathbf{P}^1, \ldots, \mathbf{P}^1).$$

For any integer k, the assignment $M \longmapsto \mathrm{pr}_{12,\star}(M \otimes \mathrm{pr}_3^\star \mathcal{O}(k))$ defines a morphism

$$m(k) : K(X, Y \times \mathbf{P}^1) \longrightarrow K(X, Y).$$

Now the difference $m - m(-1)$ descends to a morphism

$$W^{j+1} KH(X) \longrightarrow W^j KH(X),$$

defining a filtration $W^\bullet KH(X)$ on

$$KH(X) := \operatorname*{colim}_{\Delta} K(\Delta^\bullet_X).$$

3.2. — Suppose now X is regular and noetherian. Then $KH(X) \simeq K(X)$, and the filtration can be regarded as a filtration on $K(X)$ itself.

The following result will be a consequence of our main theorem, in the last section.

Theorem 3.3. — *Suppose $S = \mathrm{Spec}\, k$. Then the successive quotients can be expressed as*

$$\begin{aligned} W^{t/t+1} KH(X) &\simeq \operatorname*{colim}_{\Delta} \mathrm{cr}^j K_0(\Delta^\bullet_X; \mathbf{P}^1, \mathbf{P}^1, \ldots, \mathbf{P}^1) \\ &\simeq \operatorname*{colim}_{\Delta} \mathrm{coker}\left[\sum_{j=1}^{n} K_0\left(\Delta^\bullet_X \times (\mathbf{P}^1)^{\times(j-1)}\right) \longrightarrow K_0\left(\Delta^\bullet_X \times (\mathbf{P}^1)^{\times t}\right)\right]. \end{aligned}$$

In particular, they are simplicial $\mathbf{Z}$-modules.

Definition 3.4. — For any $j \geq 0$, let us write $\mathbf{Z}(j) := \Omega^{2j} W^{j/j+1} KH(X)$.

3.5. — The filtration $W^\bullet KH(X)$ gives rise to a spectral sequence

$$E_1^{s,t} = \pi_{s+t} W^{t/t+1} KH(X) \Longrightarrow K_{s+t}(X).$$

This is the *Atiyah-Hirzebruch spectral sequence for algebraic K-theory*. Using our $\mathbf{Z}(j)$, the E_2 page can be reindexed to take a more familiar form for geometers:

$$E_2^{s,t} = H^{s-t}(X, \mathbf{Z}(-t)) \Longrightarrow K^{s+t}(X).$$

Observe that the differentials are torsion, and so this spectral sequence degenerates rationally.

We will prove the following result and its corollaries in a later seminar.

Theorem 3.6. — *The actions of the Adams operations of $\mathbf{Z}(j)$ are pure of weight j.*

Corollary 3.7. — *The filtration on $K_*(X)$ given by the spectral sequence*

$$E_2^{s,t} = H^{s-t}(X,\mathbf{Z}(-t)) \Longrightarrow K^{s+t}(X)$$

coincides rationally with the γ-filtration on $K_*(X)$.

Corollary 3.8. — *One has*

$$H^{s-t}(X,\mathbf{Q}(-t)) \simeq K^{s+t}(X)_{\mathbf{Q}}^{(-t)},$$

as expected by Beilinson.

Proposition 3.9. — *The quotient $W^{0/j}(X)$ is the K-theory of the following symmetric monoidal* virtual Waldhausen ∞-category*: for any $\mathbf{n} \in \Delta$, denote by $\mathscr{W}_n^j(X)$ the ind-∞-category indexed on closed subschemes $Z \subset X \times (\mathbf{P}^1)^{\times j} \times \mathbf{A}_S^n$ that are finite over $X \times \mathbf{A}_S^n$ defined by*

$$\mathscr{W}_n^j(X)_Z := \mathscr{P}\!erf\left((X \times (\mathbf{P}^1)^{\times j} \times \mathbf{A}^n) - Z\right)$$

3.10. — Note that this very same definition defines a filtration on any presheaf E of spectra:

$$W^j F(X) := \operatorname*{colim}_{n\in\Delta} \operatorname*{colim}_{Z\subset X\times(\mathbf{P}^1)^{\times j}\times\mathbf{A}_S^n} E\left((X \times (\mathbf{P}^1)^{\times j} \times \mathbf{A}^n) - Z\right).$$

4. Voevodsky's slice filtration

Suppose now S a regular noetherian scheme, and abbreviate

$$\mathscr{S}\!p(\mathrm{Sm}/S) := \mathscr{S}\!p\left(\star/L_{\mathbf{A}^1}\mathscr{S}(\mathrm{Sm}/S)_{\mathrm{Nis}}\right) \qquad \text{and} \qquad \mathscr{S}\!p_{\mathbf{P}^1}(\mathrm{Sm}/S) := \mathscr{S}\!p_{\mathbf{P}^1}\left(\star/L_{\mathbf{A}^1}\mathscr{S}(\mathrm{Sm}/S)_{\mathrm{Nis}}\right)$$

Voevodsky defines the so-called *slice filtration* on $\mathscr{S}\!p_{\mathbf{P}^1}(\mathrm{Sm}/S)$, which bears some resemblance to the usual Postnikov t-structure on spectra.

Definition 4.1. — Consider the $\mathbf{P}^1$ suspension

$$\Sigma^\infty_{\mathbf{P}^1} : (S/\mathrm{Sm}/S) \longrightarrow \mathscr{S}\!p_{\mathbf{P}^1}(\mathrm{Sm}/S),$$

and denote by $\mathscr{S}\!p_{\mathbf{P}^1}(\mathrm{Sm}/S)_{\geq 0}$ the full subcategory generated by extensions and colimits by the essential image of $\Sigma^\infty_{\mathbf{P}^1}$. Now, for any $n \in \mathbf{Z}$, set

$$\mathscr{S}\!p_{\mathbf{P}^1}(\mathrm{Sm}/S)_{\geq n} := \Sigma^n_{\mathbf{P}^1}\,\mathscr{S}\!p_{\mathbf{P}^1}(\mathrm{Sm}/S)_{\geq 0}.$$

Denote by $\mathscr{S}\!p_{\mathbf{P}^1}(\mathrm{Sm}/S)_{\leq n-1}$ the full subcategory spanned by those $\mathbf{P}^1$-spectra B such that $\mathrm{Mor}(A,B) = 0$ for any $A \in \mathscr{S}\!p_{\mathbf{P}^1}(\mathrm{Sm}/S)_{\geq n}$.

Example 4.2. — The presheaf of spectra $W^n K$ on (Sm/S) is represented by a $\mathbf{P}^1$ spectrum

$$W^n\mathbf{BGL} \in \mathscr{S}\!p_{\mathbf{P}^1}(\mathrm{Sm}/k)_{\leq n}.$$

Definition 4.3. — We also have the adjunction

$$\Sigma^\infty_{\mathbf{G}_m} : \mathscr{S}\!p(\mathrm{Sm}/S) \rightleftarrows \mathscr{S}\!p_{\mathbf{P}^1}(\mathrm{Sm}/S) : \Omega^\infty_{\mathbf{G}_m}.$$

We pull back the categories $\mathscr{S}\!p_{\mathbf{P}^1}(\mathrm{Sm}/S)_{\geq n}$ along $\Sigma^\infty_{\mathbf{G}_m}$, so that $\mathscr{S}\!p(\mathrm{Sm}/S)_{/\geq n}$ is the full subcategory spanned by those spectra A such that $\Sigma^\infty_{\mathbf{G}_m}(A) \in \mathscr{S}\!p_{\mathbf{P}^1}(\mathrm{Sm}/S)_{\geq n}$. Note in particular that

$$\mathscr{S}\!p(\mathrm{Sm}/S)_{/\geq 0} = \mathscr{S}\!p(\mathrm{Sm}/S).$$

Denote by $\mathscr{S}p(\mathrm{Sm}/S)_{/\leq n-1}$ the full subcategory spanned by those spectra B such that $\mathrm{Mor}(A,B)=0$ for any $A\in\mathscr{S}p(\mathrm{Sm}/S)_{/\geq n}$.

The following is a result of a delooping machine for n-fold $\mathbf{G}_m$-loop spaces.

Lemma 4.4. — *The functor $\Omega^\infty_{\mathbf{G}_m}$ preserves the filtrations, so that*

$$\Omega^\infty_{\mathbf{G}_m}\left(\mathscr{S}p_{\mathbf{P}^1}(\mathrm{Sm}/S)_{\geq n}\right)\subset\mathscr{S}p(\mathrm{Sm}/S)_{/\geq n}$$

Lemma 4.5. — *The inclusion $\mathscr{S}p_{\mathbf{P}^1}(\mathrm{Sm}/S)_{\geq n}\hookrightarrow\mathscr{S}p_{\mathbf{P}^1}(\mathrm{Sm}/S)$ admits a right adjoint $\tau_{\geq n}$. Similarly, the inclusion $\mathscr{S}p_{\mathbf{P}^1}(\mathrm{Sm}/S)_{\leq n-1}\hookrightarrow\mathscr{S}p_{\mathbf{P}^1}(\mathrm{Sm}/S)$ admits a right adjoint*

$$\tau_{\leq n-1}=\mathrm{cofib}\left[\tau_{\geq n}\longrightarrow\mathrm{id}\right].$$

Definition 4.6. — We can use these functors to define the *slice tower*

$$\cdots\longrightarrow\tau_{\geq n+1}\longrightarrow\tau_{\geq n}\longrightarrow\tau_{\geq n-1}\longrightarrow\cdots$$

and its subquotients, the *slice functors*

$$\sigma_n=\tau_{\leq n}\tau_{\geq n}.$$

The following result will be a direct consequence of our main theorem.

Theorem 4.7. — *Suppose $S=\operatorname{Spec}k$. Then the 0-slice $\sigma_0(\mathbf{1})$ of the sphere spectrum is the motivic Eilenberg-Mac Lane spectrum $H\mathbf{Z}$.*

Corollary 4.8. — *The 0-slice $\sigma_0\mathbf{BGL}$ of $\mathbf{BGL}$ is the motivic Eilenberg-Mac Lane spectrum $H\mathbf{Z}$.*

Proof. — The unit map $\mathbf{1}\longrightarrow\mathbf{BGL}$ induces a map

$$H\mathbf{Z}=\sigma_0(\mathbf{1})\longrightarrow\sigma_0\mathbf{BGL}.$$

Since $H\mathbf{Z}\in\mathscr{S}p_{\mathbf{P}^1}(\mathrm{Sm}/k)_{\geq 0}$, it's easy to see that it suffices to show that

$$\Omega^\infty_{\mathbf{G}_m}H\mathbf{Z}\longrightarrow\Omega^\infty_{\mathbf{G}_m}\sigma_0\mathbf{BGL}$$

is an equivalence of $\mathscr{S}p(\mathrm{Sm}/k)$. Note that $H\mathbf{Z}=\Omega^\infty_{\mathbf{G}_m}H\mathbf{Z}$, since weight zero motivic cohomology is

$$H^i(X,\mathbf{Z}(0))=\begin{cases}\mathbf{Z} & \text{if } i=0\\ 0 & \text{else}\end{cases}$$

for smooth connected k-schemes.

Now we're reduced to showing that

$$\Sigma^\infty B\,\mathrm{GL}_\infty\in\mathscr{S}p(\mathrm{Sm}/S)_{\geq 1}.$$

So the claim is that for any $N\geq k\geq 0$, the spectrum $\Sigma^\infty G_S(k,N)$ lies in $\mathscr{S}p(\mathrm{Sm}/S)_{/\geq 1}$. For this, we find a divisor with normal crossings in $G_S(m,n)$ whose complement is affine N-space, and we employ homotopy purity. □

Corollary 4.9. — *The E_2 page of the spectral sequence associated to the slice filtration*

$$\cdots \longrightarrow \tau_{\geq n+1}\mathbf{BGL} \longrightarrow \tau_{\geq n}\mathbf{BGL} \longrightarrow \tau_{\geq n-1}\mathbf{BGL} \longrightarrow \cdots \longrightarrow \mathbf{BGL}$$

can be written as

$$E_2^{s,t} = H^{s-t}(X, \mathbf{Z}(-t)) \cong [\Sigma^\infty_{\mathbf{P}^1} X_+, S^s \wedge \mathbf{G}_m^{\wedge -t} \wedge \sigma_0 \mathbf{BGL}(X)] \Longrightarrow K^{s+t}(X).$$

4.10. — Note that even though the filtration on the $\mathbf{P}^1$ spectrum $\mathbf{BGL}$ is biinfinite, the induced filtration $F^\bullet K(X)$ on the spectrum $K(X)$ is finite, since

$$\pi_{q-p} F^n K(X) = [\Sigma^\infty_{\mathbf{P}^1} X_+, S^p \wedge \mathbf{G}_m^{\wedge q} \wedge \mathbf{BGL}] \simeq [\Sigma^\infty X_+, S^p \wedge \mathbf{G}_m^{\wedge q} \wedge \Omega^\infty_{\mathbf{G}_m} \tau_{\leq n} \mathbf{BGL}].$$

5. Comparison theorems

Now we wish to describe the relations among Grayson's filtration, Voevodsky's slice filtration, and the motivic Eilenberg-Mac Lane spectrum. Fix a perfect field k.

Theorem 5.1. — *The natural morphism $W^n\mathbf{BGL} \longrightarrow \tau_{\geq n}\mathbf{BGL}$ is an equivalence.*

Proof. — It's enough to find a map $\tau_{\geq n}\mathbf{BGL} \longrightarrow W^n\mathbf{BGL}$ that factors the counit $\tau_{\geq n}\mathbf{BGL} \longrightarrow \mathbf{BGL}$, and for this, it suffices to show that the composite $\tau_{\geq n}\mathbf{BGL} \longrightarrow W^{0/n}\mathbf{BGL}$ is zero.

To finish the proof, one employs a somewhat subtle geometric argument (and moving lemma) to finish the proof. □

Notation 5.2. — Recall that we have the ∞-category

$$\mathscr{M}ot(\mathrm{Sm}/k) := \mathscr{S}p_{\mathbf{P}^1}\left(L_{\mathbf{A}^1}\mathbf{Z}_{\mathrm{tr}}(\mathrm{Sm}/k)_{\mathrm{Nis}}\right)$$

of $\mathbf{P}^1$-spectra in $\mathbf{A}^1$-local presheaves with transfer on Sm/k, and we have an adjunction

$$\mathscr{F} : \mathscr{S}p_{\mathbf{P}^1}(\mathrm{Sm}/k) \rightleftarrows \mathscr{M}ot(\mathrm{Sm}/k) : \mathscr{H}.$$

We defined:

$$H\mathbf{Z} := \mathscr{H}\mathscr{F}(1).$$

Theorem 5.3. — *The slice endofunctors σ_n on $\mathscr{S}p(\mathrm{Sm}/k)$ factor as $\mathscr{H} \circ s_n$ for a functor*

$$s_n : \mathscr{S}p_{\mathbf{P}^1}(\mathrm{Sm}/k) \longrightarrow \mathscr{M}ot(\mathrm{Sm}/k).$$

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