121 EXERCISES ON LOCALLY COMPACT ABELIAN GROUPS: AN INVITATION TO HARMONIC ANALYSIS

by

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This is a collection of challenging exercises designed to motivate interested students of general topology to contemplate Pontryagin duality and the structure of locally compact abelian groups. The idea is to use the topology background students have acquired as a jumping off point to the study of (abstract) harmonic analysis.

Harmonic analysis is a towering edifice, and the story here will take us on a tour of some of the most significant theorems of the first half of the 20th century. Accordingly, these exercises become very involved and (especially in later chapters) require some fairly sophisticated background in analysis and algebra. (At times, I will simply have to appeal to certain facts, which as exercises would be absurd or heroic undertakings.) The student with limited background in these areas may wish simply to read through the remainder of these notes, and to return to them at a later date in his/her career, when the needed facts are part of his/her repertoire. In any case, any topology student (given enough time) should be able to complete the exercises from § 2.

The order of presentation here closely mirrors the beautiful two-volume treatise of Hewitt and Ross, though I have also added a number of special topics.

Contents

1. Challenges and curiosities

Example 1.1. — Try to compute the integrals

$$\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} dt \qquad \text{and} \qquad \int_0^{\infty} \frac{t^2}{(1+t^2)^2}\, dt$$

Can you manage them?

Example 1.2. — Suppose $s \in \mathbf{C}$; here's a series that depends on s:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

One can show that this series converges absolutely for $\Re(s) > 1$. The resulting function is called the *Riemann zeta function*. So what is its value at, say, $s = 2$? This question was originally known as the *Basel problem*. More generally, if m is a positive integer, can you compute

$$\zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}}?$$

Here's something even more unusual. In Srinivasa Ramanujan's second letter to G. H. Hardy [27 February 1913], he wrote:

> Dear Sir, I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich's Infinite Series and not fall into the pitfalls of divergent series. … I told him that the sum of an infinite number of terms of the series: $1+2+3+4+\cdots = -1/12$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal. I dilate on this simply to convince you that you will not be able to follow my methods of proof if I indicate the lines on which I proceed in a single letter.

He seems to have written that $\zeta(-1) = -\frac{1}{12}$. But the series above is plainly divergent when $\Re(s) \leq 1$. What on earth can he have meant?

2. Topological groups

Definition 2.1. — A *topological group* G is a topological space that is also a group if the following conditions are satisfied.

(2.1.1) The multiplication map $G \times G \longrightarrow G$, $(g,h) \longmapsto gh$, is continuous.

(2.1.2) The inverse map $G \longrightarrow G$, $g \longmapsto g^{-1}$, is continuous.

2.2. — Recall that a space X is said to be *Kolmogoroff* or T_0 if for any two distinct points $x, y \in X$, there is an open neighborhood of one that does not contain the other. For our purposes, all topological groups will be *assumed*, without further comment, to be Kolmogoroff.

Definition 2.3. — If G and G' are topological groups, then a *continuous homomorphism* $\phi : G' \longrightarrow G$ is a homomorphism of groups that is also a continuous map of topological spaces. A *topological isomorphism* $\phi : G' \longrightarrow G$ is an isomorphism of groups that is both continuous and open.

2.4. — The structure consisting of a collection of *objects* (in this case, topological groups) and *morphisms* (in this case, continuous homomorphisms) with an associative composition law is called a *category*. Most subjects in mathematics can be described as the study of a certain category or the relationship between two categories.

In these notes, there's no need to use the theory of categories in any serious way, but it is helpful to be able to refer to the *category of topological groups*, which we will denote $\mathcal{T}op\mathcal{G}p$.

Examples. — One of the many great beauties of the study of topological groups is the large number of explicit, interesting examples that one can write down.

Example 2.5. — (2.5.1) Discrete groups are topological groups.

(2.5.2) The set $\mathbf{Q}$ with the subspace topology is a topological group under addition.

(2.5.3) Any finite-dimensional vector space E over $\mathbf{R}$ can be given a topology by lifting the euclidean space topology along an isomorphism $E \cong \mathbf{R}^n$. (Check that this topology does not depend on the choice of isomorphism.) The result is a topological group under vector addition. In this case, E is said to be a *vector group*.

(2.5.4) The set of positive real numbers $\mathbf{R}_{>0}$ (with the subspace topology) is a topological group under multiplication.

(2.5.5) For any integer $n > 0$, the set

$$\mathbf{GL}_n(\mathbf{R}) := \{A \in \mathbf{Mat}_{n\times n}(\mathbf{R}) \mid \det A \neq 0\}$$

of invertible $n \times n$ matrices with real entries is a group under matrix multiplication. It is a topological group when endowed with the subspace topology from $\mathbf{Mat}_{n\times n}(\mathbf{R}) \cong \mathbf{R}^{n^2}$. It is called the *real general linear group*.

(2.5.6) For any integer $n > 0$, the set

$$\mathbf{SL}_n(\mathbf{R}) := \{A \in \mathbf{Mat}_{n\times n}(\mathbf{R}) \mid \det A = 1\}$$

of $n \times n$ matrices of determinant 1 with real entries is a group under matrix multiplication. It is a topological group when endowed with the subspace topology from $\mathbf{R}^{n^2}$. It is called the *real special linear group*.

(2.5.7) For any integer $n > 0$, the set

$$\mathbf{GL}_n(\mathbf{C}) := \{A \in \mathbf{Mat}_{n\times n}(\mathbf{C}) \mid \det A \neq 0\}$$

of invertible $n \times n$ matrices with complex entries is a group under matrix multiplication. It is a topological group when endowed with the subspace topology from $\mathbf{C}^{n^2}$. It is called the *complex general linear group*.

(2.5.8) For any integer $n > 0$, the set

$$\mathbf{SL}_n(\mathbf{C}) := \{A \in \mathbf{Mat}_{n\times n}(\mathbf{C}) \mid \det A = 1\}$$

of $n \times n$ matrices of determinant 1 with complex entries is a group under matrix multiplication. It is a topological group when endowed with the subspace topology from $\mathbf{C}^{n^2}$. It is called the *complex special linear group*.

(2.5.9) For any integer $n > 0$, the set

$$\mathbf{O}(n) := \{A \in \mathbf{Mat}_{n\times n}(\mathbf{R}) \mid A \text{ is orthogonal}\}$$

of $n \times n$ real orthogonal matrices (i.e., invertible matrices A with real coefficients such that ${}^tA = A^{-1}$) is a group under matrix multiplication. It is a topological group when endowed with the subspace topology from $\mathbf{R}^{n^2}$. It is called the *orthogonal group*.

(2.5.10) For any integer $n > 0$, the set

$$\mathbf{SO}(n) := \{A \in \mathbf{Mat}_{n\times n}(\mathbf{R}) \mid A \text{ is orthogonal and } \det A = 1\}$$

of $n \times n$ real orthogonal matrices of determinant 1 is a group under matrix multiplication. It is a topological group when endowed with the subspace topology from $\mathbf{R}^{n^2}$. It is called the *special orthogonal group*.

(2.5.11) For any integer $n > 0$, the set

$$\mathbf{Sp}_{2n}(\mathbf{R}) := \{A \in \mathbf{Mat}_{2n\times 2n}(\mathbf{R}) \mid A \text{ is symplectic}\}$$

of $2n \times 2n$ real symplectic matrices (i.e., invertible matrices A such that ${}^tA\Omega A = \Omega$, where

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

is a $2n \times 2n$ block matrix, with I_n the identity $n \times n$ matrix) is a group under matrix multiplication. It is a topological group when endowed with the subspace topology from $\mathbf{R}^{4n^2}$. It is called the *real symplectic group*.

(2.5.12) For any integer $n > 0$, the set

$$\mathbf{Sp}_{2n}(\mathbf{C}) := \{A \in \mathbf{Mat}_{2n\times 2n}(\mathbf{C}) \mid A \text{ is symplectic}\}$$

of $2n \times 2n$ complex symplectic matrices is a group under matrix multiplication. It is a topological group when endowed with the subspace topology from $\mathbf{C}^{4n^2}$. It is called the *complex symplectic group*.

(2.5.13) For any integer $n > 0$, the set

$$\mathbf{U}(n) := \{A \in \mathbf{Mat}_{n\times n}(\mathbf{C}) \mid A \text{ is unitary}\}$$

of $n \times n$ unitary matrices (i.e., invertible matrices A such that ${}^t\overline{A} = A^{-1}$) is a group under matrix multiplication. It is a topological group when endowed with the subspace topology from $\mathbf{C}^{n^2}$. It is called the *unitary group*.

(2.5.14) For any integer $n > 0$, the set

$$\mathbf{SU}(n) := \{A \in \mathbf{Mat}_{n\times n}(\mathbf{R}) \mid A \text{ is unitary and } \det A = 1\}$$

of $n \times n$ unitary matrices of determinant 1 is a group under matrix multiplication. It is a topological group when endowed with the subspace topology from $\mathbf{C}^{n^2}$. It is called the *special unitary group*.

(2.5.15) Finally, there is one additional topological group worth mentioning here, namely

$$\mathbf{Sp}(n) := \mathbf{U}(2n) \cap \mathbf{Sp}_{2n}(\mathbf{C}).$$

It is sometimes called the *hyperunitary group*, but we will call it the *compact symplectic group*, for reasons that will quickly become clear.

Exercise 2.6. — Show that the topological groups $\mathbf{O}(n)$, $\mathbf{SO}(n)$, $\mathbf{U}(n)$, $\mathbf{SU}(n)$, and $\mathbf{Sp}(n)$ are all compact. What about the others?

Exercise 2.7. — Show that the topological groups $\mathbf{GL}_n(\mathbf{R})$, $\mathbf{SL}_n(\mathbf{R})$, $\mathbf{GL}_n(\mathbf{C})$, $\mathbf{SL}_n(\mathbf{C})$, $\mathbf{O}(n)$, $\mathbf{SO}(n)$, $\mathbf{Sp}_{2n}(\mathbf{R})$, $\mathbf{Sp}_{2n}(\mathbf{C})$, $\mathbf{U}(n)$, $\mathbf{SU}(n)$, and $\mathbf{Sp}_{2n}(\mathbf{C})$ are all *locally euclidean*, in the sense that they each possess a neighborhood of the identity that is homeomorphic to an open ball in euclidean space of dimension n^2, n^2-1, $2n^2$, $2n^2-2$, $n(n-1)/2$, $n(n-1)/2$, $n(2n+1)$, $2n(2n+1)$, n^2, n^2-1, and $n(2n+1)$ respectively. (In particular, each of these groups is locally compact.) This particular collection of topological groups are sometimes called the *linear groups*.

To demonstrate this, introduce the exponential of a matrix $A \in \mathbf{Mat}_{d\times d}(\mathbf{C}) \cong \mathbf{C}^{d^2}$ in the following manner. Observe that if $A = (a_{ij})$, and $\alpha = \sup\{a_{ij}\}_{i,j}$, then no element of A^ℓ has absolute value greater than $(d\alpha)^\ell$. Use this to verify that the sequence $(\exp_m)_{m\geq 0}$ of continuous functions $\mathbf{Mat}_{d\times d}(\mathbf{C}) \longrightarrow \mathbf{Mat}_{d\times d}(\mathbf{C})$ given by the formula

$$\exp_m(A) := \sum_{\ell=0}^{m} \frac{1}{\ell!} A^\ell$$

converges for the compact-open topology on $\mathscr{C}(\mathbf{Mat}_{d\times d}(\mathbf{C}), \mathbf{Mat}_{d\times d}(\mathbf{C}))$. Now define exp as the limit, so that

$$\exp(A) = e^A := \sum_{\ell=0}^{\infty} \frac{1}{\ell!} A^\ell$$

for any $A \in \mathbf{Mat}_{n\times n}(\mathbf{C})$.

Now verify the following facts about the exponential.

(2.7.1) For any $A \in \mathbf{Mat}_{d\times d}(\mathbf{C})$, the matrix e^A is invertible.

(2.7.2) For any $A \in \mathbf{Mat}_{d\times d}(\mathbf{C})$ and $B \in \mathbf{GL}_n(\mathbf{C})$, one has $e^{B^{-1}AB} = B^{-1}e^A B$.

(2.7.3) If $A \in \mathbf{Mat}_{d\times d}(\mathbf{C})$ has eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_d$, then e^A has eigenvalues $e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_d}$.

(2.7.4) For any $A \in \mathbf{Mat}_{d\times d}(\mathbf{C})$, one has $\det e^A = e^{\operatorname{tr} A}$.

(2.7.5) If $A, B \in \mathbf{Mat}_{d\times d}(\mathbf{C})$, and $AB = BA$, then $e^{A+B} = e^A e^B$.

(2.7.6) For any $A \in \mathbf{Mat}_{d\times d}(\mathbf{C})$, one has $e^{{}^tA} = {}^t(e^A)$.

(2.7.7) For any $A \in \mathbf{Mat}_{d\times d}(\mathbf{C})$, one has $e^{\overline{A}} = \overline{(e^A)}$.

Use this exponential and these facts to give a homeomorphism to a neighborhood of the identity in each of these linear groups from an open ball in euclidean space of the appropriate dimension.

2.8. — Let us summarize what we have learned about our linear groups in the following table.

Linear group	Name	Compact?	Dimension
$\mathbf{GL}_n(\mathbf{R})$	real general linear group	no	n^2
$\mathbf{SL}_n(\mathbf{R})$	real special linear group	no	n^2-1
$\mathbf{GL}_n(\mathbf{C})$	complex general linear group	no	$2n^2$
$\mathbf{SL}_n(\mathbf{C})$	complex special linear group	no	$2n^2-2$
$\mathbf{O}(n)$	orthogonal group	yes	$n(n-1)/2$
$\mathbf{SO}(n)$	special orthogonal group	yes	$n(n-1)/2$
$\mathbf{Sp}_{2n}(\mathbf{R})$	real symplectic group	no	$n(2n+1)$
$\mathbf{Sp}_{2n}(\mathbf{C})$	complex symplectic group	no	$2n(2n+1)$
$\mathbf{U}(n)$	unitary group	yes	n^2
$\mathbf{SU}(n)$	special unitary group	yes	n^2-1
$\mathbf{Sp}(n)$	compact symplectic group	yes	$n(2n+1)$

Exercise 2.9. — Show that $\mathbf{U}(1)$ is topologically isomorphic to the group of complex numbers of norm 1 under the usual multiplication in $\mathbf{C}$, with the usual subspace topology. Hence the assignment $\theta \longmapsto e^{2\pi\theta\sqrt{-1}}$ permits us to identify $\mathbf{U}(1)$ topologically with the quotient $[0,1]/\{0,1\}$.

Exercise 2.10. — Show that $\mathbf{SO}(3)$ can be interpreted as the group of all possible rotations in $\mathbf{R}^3$. In particular, for any $\phi \in [0,2\pi]$, and any $\theta \in [0,\pi]$, we have

$$X(\phi) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \qquad \text{and} \qquad Z(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the rotations around the x- and z-axes. Show that any element of $\mathbf{SO}(3)$ can be written uniquely as a product $Z(\psi)X(\theta)Z(\phi)$, for $\phi,\psi \in [0,2\pi]$ and $\theta \in [0,\pi]$. The angles ϕ,ψ,θ are known as *Euler angles*.

Exercise 2.11. — More generally, show that for any positive integer n, the group $\mathbf{SO}(n)$ acts transitively on the sphere S^{n-1}.

The topology of groups. — One of the main themes of the study of topological groups is the insight that the topology of a topological group is completely controlled by small neighborhoods of the identity.

Exercise 2.12. — Suppose G is a group with a T_1 topology (so that any point is closed in G). Suppose that the map $G \times G \longrightarrow G$, $(g,h) \longmapsto g^{-1}h$, is continuous. Show that G is a topological group.

Exercise 2.13. — Suppose G a topological group with identity element $e \in G$, and suppose $\mathscr{B}_e$ a fundamental system of open neighborhoods of e. Then verify that the set

$$\{gU \mid g \in G, U \in \mathscr{U}\} \cup \{Ug \mid g \in G, U \in \mathscr{U}\}$$

is a base for the topology of G.

Definition 2.14. — Suppose G a topological group. Then a family $\mathscr{A}$ of neighborhoods of the identity element $e \in G$ is said to be *coherent* if it satisfies the following properties:

(2.14.1) For every $U \in \mathscr{A}$, there exists a $V \in \mathscr{A}$ such that $V^2 \subset U$.
(2.14.2) For every $U \in \mathscr{A}$, there exists a $V \in \mathscr{A}$ such that $V^{-1} \subset U$.

One says that a coherent family $\mathscr{A}$ is *translatable* if, moreover,

(2.14.3) For every $U \in \mathscr{A}$ and every $g \in U$, there exists a $V \in \mathscr{A}$ such that $gV \subset U$.

One says that a coherent family $\mathscr{A}$ is *normalizable* if, moreover,

(2.14.4) For every $U \in \mathscr{A}$ and every $g \in G$, there exists a $V \in \mathscr{B}_e$ such that $gVg^{-1} \subset U$.

Exercise 2.15. — Suppose G a topological group, and suppose $\mathscr{B}_e$ a fundamental system of open neighborhoods of e. Then check that $\mathscr{B}_e$ is a translatable, normalizable, coherent family.

Conversely, suppose G a group, and suppose $\mathscr{A}$ a normalizable, translatable, coherent family of open neighborhoods of e with the finite intersection property. Show that the set $\{gU \mid g \in G, U \in \mathscr{A}\}$ (or, equivalently, the set $\{gU \mid g \in G, U \in \mathscr{A}\}$) is a subbase for a topology on G relative to which G is a topological group.

Exercise 2.16. — Suppose G is a topological group, and suppose A and B subsets of G. Prove the following.

(2.16.1) If A is open, then AB and BA are open.
(2.16.2) If A and B are compact, then AB and BA are compact.
(2.16.3) If A is closed and B is compact, then AB and BA are closed.
(2.16.4) $\left(\overline{A}\right)\left(\overline{B}\right) \subset \overline{AB}$.
(2.16.5) $\left(\overline{A}\right)^{-1} = \overline{A^{-1}}$.
(2.16.6) $g\overline{A}h = \overline{gAh}$ for any $g, h \in G$.
(2.16.7) If $ab = ba$ for any $a \in A$ and $b \in B$, then $ab = ba$ for any $a \in \overline{A}$ and $b \in \overline{B}$.

Example 2.17. — If A and B are closed, it does not follow that AB is closed. In $\mathbf{R}$, consider the closed sets $\mathbf{Z}$ and $\alpha\mathbf{Z}$, where α is an irrational number. Then $\mathbf{Z} + \alpha\mathbf{Z}$ is dense and not closed in $\mathbf{R}$.

Exercise 2.18. — Show that every topological group G has a fundamental system of open neighborhoods $\mathscr{B}_e$ at e such that any $U \in \mathscr{B}_e$ is *symmetric* in the sense that one has $U = U^{-1}$.

Exercise 2.19. — Use what you have shown so far to demonstrate that every topological group G (only assumed to be Kolmogoroff!) is *regular* or T_3.(1) In particular, G is Hausdorff (or T_2).

Theorem 2.20. — *Suppose G a topological group, suppose U any neighborhood of e, and suppose $K \subset G$ any compact subset. Then there is a neighborhood V of e such that for any $g \in K$, one has $gVg^{-1} \subset U$.*

Proof. — Let $\mathscr{U}_e$ be the collection of all neighborhoods U of e such that $U = U^{-1}$. It is enough to show that for any element $h \in G$, there is an element $V \in \mathscr{U}_e$ such that for any $g \in Vh$, one has $gVg^{-1} \subset U$. (Why is this enough? Hint: Use the compactness of K.)

Using the exercises above, we may find $W \in \mathscr{U}_e$ such that $W^3 \subset U$ and $W' \in \mathscr{U}_e$ such that $hW'h^{-1} \subset W$. Now set $V = W \cap W'$; then for any $g \in Vh$, one has $gh^{-1} \in W$ and thus $hg^{-1} \in W$, whence

$$gVg^{-1} \subset gW'g^{-1} = gh^{-1}gW'h^{-1}hg^{-1} \subset W^3 \subset U,$$

as desired. □

Subgroups and quotients of a topological group. — One wishes to see that topological groups are closed under the usual operations of group theory. We begin with subgroups and quotient groups It will be especially important to analyze under what circumstances a subgroup or quotient group inherits good properties from a give topological group.

Exercise 2.21. — Suppose G a topological group. Show that if H is a subgroup, normal subgroup, or abelian subgroup of G, then so is $\overline{H}$.

Exercise 2.22. — Show that a subgroup H of a topological group G is open if and only if its interior is nonempty; if H is open, then it is also closed.

Exercise 2.23. — Show that the subgroup of a topological group generated by a symmetric neighborhood of the identity is open (hence closed).

(1)A space X is said to be regular if it is T_1 and for any closed set $C \subset G$ and every point $g \in G$ not contained in C, there exist disjoint open sets containing C and g, respectively

Exercise 2.24. — Suppose G a topological group, and suppose $\mathscr{A}$ a coherent family of open neighborhoods of e such that for any elements $U, V \in \mathscr{A}$, there exists an element $W \in \mathscr{A}$ such that $W \subset U \cap V$. Show the set

$$H := \bigcap_{U \in \mathscr{A}} U$$

is a closed subgroup of G, which is normal if $\mathscr{A}$ is normalizable.

Exercise 2.25. — Suppose G a topological group, H a subgroup of G. Show that H is closed in G if and only if there is a neighborhood U of e such that $\overline{U} \cap H$ is closed. Conclude that if H is locally compact (in the subspace topology), then H is closed.

Exercise 2.26. — Show that a subgroup H of a topological group G is discrete if and only if it has an isolated point. If H is discrete, check that H is closed.

Exercise 2.27. — Suppose G a locally compact topological group. Show that the following are equivalent.

(2.27.1) There is a compact subspace $F \subset G$ that generates G.
(2.27.2) There is an open subset $U \subset G$ that generates G such that the closure $\overline{U}$ is compact.
(2.27.3) There is a neighborhood V of e that generates G such that the closure $\overline{V}$ is compact.

A locally compact topological group satisfying any, and hence all, of the above conditions is said to be *compactly generated*.(2)

Example 2.28. — A discrete group is compactly generated if and only if it is finitely generated.

Exercise 2.29. — Suppose G a topological group, and suppose $F \subset G$ a compact subspace. Show that there is an open, closed, compactly generated subgroup H of G containing F.

Definition 2.30. — Suppose G a topological group, and suppose H a subgroup of G. Then the group-theoretic quotient G/H (i.e., the set of left cosets gH for $g \in G$) can be topologized in the following manner:

$$\mathscr{O}p(G/H) := \{U \subset G/H \mid \phi^{-1}(U) \in \mathscr{O}p(G)\},$$

where $\phi : G \longrightarrow G/H$ is the usual quotient map.(3)

Exercise 2.31. — Suppose G a topological group, and suppose H a subgroup of G. Show that the quotient map $\phi : G \longrightarrow G/H$ is open and continuous, and if H is compact, then ϕ is also closed.

Exercise 2.32. — Suppose G a topological group, and suppose H a subgroup of G. Show that H is open if and only if G/H is discrete. Show that the following are equivalent.

(2.32.1) The space G/H is Kolmogoroff.
(2.32.2) The subgroup H is closed.
(2.32.3) The space G/H is regular (T_3).

Exercise 2.33. — Suppose G a compact (respectively, locally compact) topological group, and suppose H a subgroup of G. Show that G/H is compact (resp., locally compact).

Lemma 2.34. — *Suppose G a topological group and suppose H a subgroup of G. Let $\phi : G \longrightarrow G/H$ the quotient map. Suppose we are given the following additional pieces of data:*

(2.34.1) a symmetric neighborhood V of e such that $\overline{\left(\overline{V}^3\right)} \cap H$ is compact, and
(2.34.2) a subset $W \subset G$ such that $\{wH \mid w \in W\} \subset G/H$ is compact, and $\{wH \mid w \in W\} \subset \{vH \mid v \in V\}$.

Then the subspace $\overline{V} \cap WH \subset G$ is compact.

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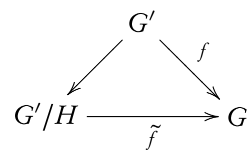
(2)This is not to be confused with the inherently topological notion of *compactly generated space*, which plays a key role in modern homotopy theory.

(3)Warning: the space G/H is not a group unless H is normal in G, and the space G/H almost never coincides with the identification space obtained by identifying all points of H to a single point.

Exercise 2.35. — Suppose G a topological group, and suppose H a subgroup of G. Show that if both H and G/H are compact (respectively, locally compact), then so is G itself. [Hint: for the "compact" statement, apply the previous lemma to $V = W = G$.]

Exercise 2.36. — Suppose G a topological group, and suppose H a normal subgroup of G. Show G/H is a topological group (possibly not Kolmogoroff). Check that G/H is Kolmogoroff (hence regular) if and only if H is closed.

Exercise 2.37. — Suppose G and G' two topological groups, and suppose $f : G' \longrightarrow G$ an open, continuous, surjective homomorphism. Then $\ker f$ is a closed, normal subgroup of G', and there is a unique topological isomorphism $\widetilde{f} : G'/H \longrightarrow G$ such that the diagram



commutes.

Lemma 2.38. — *Suppose G' is a locally compact, σ-compact topological group, and suppose G a locally countably compact topological group.*[4] *Then every continuous epimorphism $f : G' \longrightarrow G$ is open.*

Exercise 2.39. — Suppose G a topological, suppose A a subgroup of G, and suppose H a normal subgroup of G. Suppose that A is locally compact and σ-compact and that AH is locally compact as well. Show that the canonical homomorphism

$$(AH)/H \longrightarrow A/(A \cap H)$$

is a topological isomorphism.

Example 2.40. — The conditions of the previous exercise are, unfortunately, necessary. Suppose $a \in \mathbf{R}$ an irrational number. Then $a\mathbf{Z}/(a\mathbf{Z} \cap \mathbf{Z})$ is discrete, while $(a\mathbf{Z} + \mathbf{Z})/\mathbf{Z}$ is not discrete.

Exercise 2.41. — Suppose G a topological group, and suppose H and K normal subgroups such that $K \subset H$. Then the canonical homomorphism

$$G/H \longrightarrow (G/K)/(H/K)$$

is a topological isomorphism.

Products and limits. — Also important are the operations of product, restricted product, and limit. We study these here.

Definition 2.42. — Suppose $\mathscr{G} = \{G_\alpha \mid \alpha \in A\}$ a family of topological groups. Then the *product group* $\prod_{G \in \mathscr{G}} G$ is a topological group as well, equipped with the product topology. Contained therein is the subspace

$$\prod_{G \in \mathscr{G}} G := \{(g_\alpha)_{\alpha \in A} \mid \text{for all but finitely many } \alpha \in A,\ g_\alpha = e\} \subset \prod_{G \in \mathscr{G}} G.$$

This is known as the *restricted product*.

More generally, if $H_\alpha \subset G_\alpha$ is a closed subgroup for every $\alpha \in \Lambda$, then one may form the subspace

$$^{\{H_\alpha\}}\prod_{G \in \mathscr{G}} G := \{(g_\alpha)_{\alpha \in A} \mid \text{for all but finitely many } \alpha \in A,\ g_\alpha \in H_\alpha\} \subset \prod_{G \in \mathscr{G}} G.$$

Exercise 2.43. — Suppose $\mathscr{G} = \{G_\alpha \mid \alpha \in A\}$ a family of topological groups. Show that $\prod_{G \in \mathscr{G}} G$ is a topological group, and the restricted product $\coprod_{G \in \mathscr{G}} G \subset \prod_{G \in \mathscr{G}} G$ is a dense subgroup. Show that $\prod_{G \in \mathscr{G}} G$ and $\coprod_{G \in \mathscr{G}} G \subset \prod_{G \in \mathscr{G}} G$ are locally compact if each G_α is.

Moreover, show that if $H_\alpha \subset G_\alpha$ is a closed subgroup for every $\alpha \in \Lambda$, then the restricted product $^{H_\alpha}\coprod_{G \in \mathscr{G}} G \subset \prod_{G \in \mathscr{G}} G$ is a dense subgroup, which is locally compact if each G_α is locally compact, and each H_α is compact.

[4]Recall that a space X is said to be *σ-compact* if it can be written as the union of countably many compact spaces, and a space X is said to be locally countably compact if every point has a countably compact neighborhood

Exercise 2.44. — Suppose $\mathscr{G} = \{G_\alpha \mid \alpha \in A\}$ a family of topological groups, and suppose, for any $\alpha \in A$, that H_α is a subgroup of G_α. Show that the canonical homomorphism

$$\prod_{\alpha\in A}(G_\alpha/H_\alpha) \longrightarrow (\prod_{\alpha\in A} G_\alpha) \,/\, (\prod_{\alpha\in A} H_\alpha)$$

is a topological isomorphism.

Exercise 2.45. — Suppose G a topological group, and suppose $\{N_i\}_{i=1}^m$ a finite set of normal subgroups of G with the following properties;

(2.45.1) $N_1 N_2 \cdots N_m = G$;
(2.45.2) for any $k \in \{1, 2, \dots, m-1\}$, the intersection $(N_1 N_2 \cdots N_k) \cap N_{k+1} = 1$;
(2.45.3) the product $U_1 U_2 \cdots U_m$ of any neighborhoods U_i of e in N_i (for $i \in \{1, 2, \dots, m\}$) is a neighborhood of e in X.

Then the canonical homomorphism

$$\prod_{i=1}^m N_i \longrightarrow G$$

is a topological isomorphism.

Exercise 2.46. — Suppose G a locally countably compact topological group, and suppose $\{N_i\}_{i=1}^m$ a finite set of locally compact, σ-compact normal subgroups of G satisfying conditions (2.45.1-2). Use 2.38 to verify the remaining condition of the previous exercise, and deduce that the canonical homomorphism

$$\prod_{i=1}^m N_i \longrightarrow G$$

is a topological isomorphism.

Definition 2.47. — Suppose Λ a poset, and suppose $G : \Lambda^{\mathrm{op}} \longrightarrow \mathscr{T}\!op\mathscr{G}p$ a *functor*. That is, we are given the following data:

(2.47.1) for any element $\alpha \in \Lambda$, a topological group G_α, and
(2.47.2) for any $\alpha \leq \beta$, a continuous homomorphism

$$\phi_{\beta\alpha} : G_\beta \longrightarrow G_\alpha,$$

subject to the following axioms:

(2.47.3) $\phi_{\alpha\alpha} = \mathrm{id}_{G_\alpha}$, and
(2.47.4) $\phi_{\gamma\alpha} = \phi_{\beta\alpha} \circ \phi_{\gamma\beta}$ for any $\alpha \leq \beta \leq \gamma$.

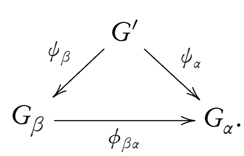
This is sometimes called an *inverse system* of topological groups (though the term *functor* strikes me as substantially clearer).

The *limit* (sometimes called the *inverse limit* or *projective limit*) of the functor G is the subspace

$$\lim G = \lim_{\alpha\in\Lambda^{\mathrm{op}}} G_\alpha := \{(g_\alpha)_{\alpha\in\Lambda} \mid \text{for any } \alpha \leq \beta,\ g_\alpha = \phi_{\beta\alpha}(g_\beta)\} \subset \prod_{\alpha\in\Lambda} G_\alpha.$$

Exercise 2.48. — For any functor $G : \Lambda^{\mathrm{op}} \longrightarrow \mathscr{T}\!op\mathscr{G}p$, show that the limit $\lim G \subset \prod_{\alpha\in\Lambda} G_\alpha$ is a closed subgroup.

Suppose G' a topological group. Then a collection $\{\psi_\alpha\}_{\alpha\in\Lambda}$ of topological homomorphisms $\psi_\alpha : G' \longrightarrow G_\alpha$ is said to be a *compatible system of topological homomorphisms* if, for any $\alpha \leq \beta$, the following diagram commutes:



Verify that the projection maps

$$\mathrm{pr}_\alpha : \lim G \subset \prod_{\alpha\in\Lambda} G_\alpha \longrightarrow G_\alpha$$

provide a compatible system $\{\mathrm{pr}_\alpha\}_{\alpha\in\Lambda}$ of topological homomorphisms $\lim G \longrightarrow G_\alpha$. Moreover, verify that $\{\mathrm{pr}_\alpha\}_{\alpha\in\Lambda}$ is *universal* in the following sense: for any compatible system $\{\psi_\alpha\}_{\alpha\in\Lambda}$ of topological homomorphisms $\psi_\alpha : G' \longrightarrow G_\alpha$, there exists a unique topological homomorphism $\psi : G' \longrightarrow \lim G$ with the property that for any $\alpha \in \Lambda$, the following diagram commutes:

$$\begin{array}{ccc} & G' & \\ {}^{\psi}\swarrow & & \searrow^{\psi_\alpha} \\ \lim G & \xrightarrow[\mathrm{pr}_\alpha]{} & G_\alpha . \end{array}$$

Further topological properties of topological groups. — It is frequently very helpful to understand the nature of the connectedness of a topological group.

Notation 2.49. — Suppose G a topological group. Then the connected component of G containing the identity element e will be denoted G_0.

Exercise 2.50. — For any topological group G, the connected component $G_0 \subset G$ is a closed normal subgroup. The quotient group G/G_0 is totally disconnected.[5]

Exercise 2.51. — Suppose G a topological group, and suppose U a neighborhood of e. Use 2.23 to show that G_0 is contained in the group generated by U.

Exercise 2.52. — Suppose G a topological group. Any compact neighborhood U of e contains a subgroup that both compact and open. If G is itself compact, then U contains an open (hence closed) *normal* subgroup N, and G/N is finite.

Exercise 2.53. — Suppose G a topological group. Show that G_0 is the intersection of all open subgroups of G.

Exercise 2.54. — Show that a locally compact group G is compactly generated if G/G_0 is compact.

Exercise 2.55. — Use what you have shown so far in this subsection to show that the following are equivalent for a locally compact group G.

(2.55.1) G is connected.
(2.55.2) There is a connected subgroup H of G such that G/H is connected.
(2.55.3) G contains no proper open subgroups.
(2.55.4) Every neighborhood of e generates G.

Exercise 2.56. — Suppose G' and G two topological groups, and assume that G' is locally compact. Show that the inverse image of G_0 under any open continuous epimorphism $G' \longrightarrow G$ is G'_0.

Theorem 2.57 **(Paracompactness of locally compact groups)**. — *Every locally compact group is paracompact, hence normal.*[6]

Exercise 2.58. — Verify that the following are equivalent for a topological group G.

(2.58.1) G can be written as the limit of finite discrete groups.
(2.58.2) G is compact and the unit $e \in G$ has a fundamental system of neighborhood comprised of open and closed normal subgroups.
(2.58.3) G is compact and $G_0 = 1$.
(2.58.4) G is compact and totally disconnected (i.e., the connected components of G are one-point spaces).

A topological group G satisfying any (and hence all) of these conditions is called *profinite*.

Example 2.59. — Consider the set $\mathbf{Z}_{>0}$ of positive integers, ordered by division. Then there is a functor $\mathbf{Z}_{>0}^{\mathrm{op}} \longrightarrow \mathcal{T}op\mathcal{G}p$, which to any positive integer m assigns the finite discrete group $\mathbf{Z}/m$, and for any positive integers m and n with $m|n$, the canonical projection $\mathbf{Z}/n \longrightarrow \mathbf{Z}/m$. We may therefore form the limit:

$$\widehat{\mathbf{Z}} := \lim_{m} \mathbf{Z}/m.$$

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[5] A space is said to be *totally disconnected* if its connected components are one-point spaces.

[6] A space X is said to be *paracompact* if for every open cover $\mathcal{U}$ of X, there is a locally finite open refinement of $\mathcal{U}$. A space is said to be *normal* (or T_4) if it is T_1 and for any two closed subsets $C, D \subset X$ there exist disjoint open sets containing C and D, respectively.

[Unfortunately, there is some potential for confusion here with the notation for Pontryagin duality. Luckily, the Pontryagin dual of $\widehat{\mathbf{Z}}$ already has a nice name, so there will be no notational conflict.] This is the *profinite completion* of $\mathbf{Z}$.

Example 2.60. — Suppose p is a prime number; we now consider a "p-local" version of the above construction. We can consider the set $\mathbf{Z}_{>0}$ ordered in the usual fashion, and one may define the functor $\mathbf{Z}_{>0}^{\mathrm{op}} \longrightarrow \mathcal{T}op\mathcal{G}p$, which to andy positive integer m assigns the finite discrete group $\mathbf{Z}/p^m$, and for any $m \leq n$, there is a projection $\mathbf{Z}/p^n \longrightarrow \mathbf{Z}/p^m$. Hence we form the limit:

$$\mathbf{Z}_p := \lim_m \mathbf{Z}/p^m.$$

This is a profinite group, known as the p-adic integers.

Example 2.61. — For any positive integer n, the set $\mathbf{Z}/n$ comes equipped with a product as well. We can therefore speak of the set $(\mathbf{Z}/n)^\times$ of *units* in $\mathbf{Z}/n$, i.e., elements $a \in \mathbf{Z}/n$ such that there exists an element $a^{-1} \in \mathbf{Z}/n$ for which $aa^{-1} = 1$. These too yield a functor $\mathbf{Z}_{>0}^{\mathrm{op}} \longrightarrow \mathcal{T}op\mathcal{G}p$, where $\mathbf{Z}_{>0}$ is ordered by division. We may therefore form this limit:

$$\widehat{Z}^\times := \lim_m (\mathbf{Z}/m)^\times.$$

Similarly, for any prime number p, one may form the limit:

$$\mathbf{Z}_p^\times := \lim(\mathbf{Z}/p^m)^\times.$$

***Example 2.62* (For students with experience in field theory).** — Suppose k a field, and suppose k^{alg} a fixed algebraic closure. Then the set of all finite Galois extensions $k \subset \ell$ form an inverse system of finite groups $\mathrm{Gal}(\ell : k) := \mathrm{Aut}_k(\ell)$, where, if $\ell' \subset \ell$, then $\mathrm{Gal}(\ell : k) \longrightarrow \mathrm{Gal}(\ell' : k)$ is simply the restriction. Thus one may form the limit:

$$G_k = \mathrm{Gal}(k^{\mathrm{alg}} : k) = \lim_{k \subset \ell} \mathrm{Gal}(\ell : k).$$

This is manifestly a profinite group.

Structure theory of locally compact abelian groups. — We now show how to reduce the study of locally compact abelian (LCA) groups to compact abelian groups and discrete groups.

Exercise 2.63. — Recall that every LCA group A contains an open compactly generated subgroup H. Show that A can be written as the union of its compactly generated open subgroups. Hence our structure theory focuses on the compactly generated case, to which we now turn.

Definition 2.64. — Suppose G a topological group. Then an element $g \in G$ specifies a unique homomorphism $\mathbf{Z} \longrightarrow G$ (automatically continuous) under which $g \longmapsto 1$. The image of this homomorphism (the subgroup generated by g) is called a *cyclic subgroup* of G. If G contains a dense cyclic subgroup then G is said to be *monothetic*.

Similarly, the image of a continuous homomorphism $\mathbf{R} \longrightarrow G$ is called a *one-parameter subgroup* of G. If G contains a dense one-parameter subgroup, then G is said to be *solenoidal*.

***Exercise 2.65* (Weil's lemma).** — Suppose G a monothetic locally compact group. Show that either G is compact, or else the corresponding homomorphism $\mathbf{Z} \longrightarrow G$ is a topological isomorphism.

Exercise 2.66. — Suppose G a solenoidal locally compact group. Show that either G is compact, or else the corresponding continuous homomorphism $\mathbf{R} \longrightarrow G$ is a topological isomorphism.

Exercise 2.67. — Suppose A a compactly generated, LCA group. Then A contains a finitely generated discrete subgroup N such that A/N is compact. [Hint: Find a symmetric neighborhood U of e and a finite collection of elements $\{a_i\}_{i=1}^n$ generating a subgroup H such that $A = HU$. Now show that some of the a_i's generated the desired discrete subgroup.]

Definition 2.68. — Suppose G a topological group. An element $g \in G$ is said to be *compact* if the closure of the cyclic subgroup it generates is compact. Denote by $c(G) \subset G$ the subset of compact elements.

Now assume A an abelian topological group; let $\mathscr{F}(A)$ be the set of all finitely generated, free abelian subgroups W of A such that there is a compact subset $K \subset A$ such that $A = KW$, and $W \cap c(A) = \{e\}$. For any nonnegative integer r, let $\mathscr{F}_{\leq r}(A) \subset \mathscr{F}(A)$ be the subset of subgroups $W \in \mathscr{F}(A)$ of rank $\leq r$. Denote by $m(A) = \inf\{\operatorname{rk} W \mid W \in \mathscr{F}(A)\}$.

Exercise 2.69. — Suppose G a locally compact group. Show that $c(G)$ is a subgroup, and it contains any compact normal subgroup.

Exercise 2.70. — Use the structure theorem for finitely generated abelian groups to show that for any compactly generated LCA group A, $\mathscr{F}(A)$ is nonempty. Use Weil's lemma to deduce that every subgroup $W \in \mathscr{F}_{\leq m(A)}(A)$ is discrete, and the quotient A/W is compact.

Exercise 2.71. — Suppose A a compactly generated LCA group that contains an open compact subgroup. Use the structure theorem for finitely generated abelian groups to show that A has a maximal compact subgroup K such that $A \cong K \times W$, where $W \in \mathscr{F}_{\leq m(A)}(A)$.

Exercise 2.72. — Show that every compactly generated LCA group A is topologically isomorphic to $A' \times E$, where A' is a compactly generated LCA group with an open compact subgroup, and E is a vector group.

***Theorem 2.73* (Structure theorem for compactly generated LCA groups).** — *Every compactly generated LCA group A is topologically isomorphic to $K \times E \times \mathbf{Z}^{m(A)}$, where K is a compact abelian group and E is a vector group.*

Proof. — Immediate from the two exercises above. □

Theorem 2.74. — *Any connected LCA group A is topologically isomorphic to $K \times E$, where E is a vector group, and $K \subset A$ is a connected, maximal compact subgroup of A.*

Proof. — Also immediate. □

More examples. — We finish this section with some further examples of topological groups.

Exercise 2.75. — Show that $\mathbf{R}/\mathbf{Z}$ is topologically isomorphic to $\mathbf{U}(1)$. We write $\mathbf{T}^n = (\mathbf{R}/\mathbf{Z})^n$ for any nonnegative integer n. These LCA groups are called *tori*.

Exercise 2.76. — Show that, for any positive integer n, the group $\mathbf{SO}(n)$ contains a closed subgroup that is topologically isomorphic to $\mathbf{SO}(n-1)$ for which the quotient $\mathbf{SO}(n)/\mathbf{SO}(n-1)$ is topologically isomorphic to S^{n-1}.

Exercise 2.77. — Show that $\mathbf{SO}(2)$ and $\mathbf{U}(1)$ are topologically isomorphic.

Exercise 2.78. — Show that $\mathbf{Sp}(1)$ and $\mathbf{SU}(2)$ are topologically isomorphic.

***Exercise 2.79* (For students with experience with covering spaces).** — For any positive integer n, show that the group $\mathbf{SO}(n)$ admits a two-fold covering homomorphism $\mathbf{Spin}(n) \longrightarrow \mathbf{SO}(n)$, where $\mathbf{Spin}(n)$ is a locally compact group. The kernel of this continuous homomorphism is thus $\mathbf{Z}/2$.

***Exercise 2.80* (For students with experience with field theory).** — Suppose K an algebraic number field. A place v of K is an equivalence class of nontrivial absolute values $|\cdot|_v$ on K. Each place v gives K the structure of a metric space; one can complete K with respect to $|\cdot|_v$ to get a complete metric space K_v. Show that each K_v is an LCA group (under addition).

A place v is said to be *finite* if the resulting metric on K is an *ultrametric* in the sense that $|a+b|_v \leq \sup\{|a|_v, |b|_v\}$; otherwise it is said to be *infinite*. Write $\mathscr{P}$ for the set of finite places of K. For any finite place v, consider the subspace $\mathscr{O}_v \subset K_v$ comprised of elements $x \in K_v$ such that $|x|_v \leq 1$. This is the *ring of integers* in K_v. Show that $\mathscr{O}_v$ is a compact subgroup of K_v for every $v \in \mathscr{P}$.

Define now the *adèles* of the field K:

$$\mathbf{A}_K := {}^{\{\mathscr{O}_v\}}\prod_{v \in \mathscr{P}} K_v.$$

One can embed $K \subset \mathbf{A}_K$ by $x \longmapsto (x, x, x, \dots)$ as a closed subgroup, since for any $x \in K$, the absolute value $x \in \mathscr{O}_v$ for all but finitely many places. Consequently, one may form $\mathbf{A}_K/K$. This object plays an important role in modern number theory.

3. Haar measure

Warning 3.1. — From this point on, we must assume a background in measure theory, integration, and elementary functional analysis.

Notation 3.2. — Suppose X a locally compact Hausdorff space. Let us fix the following notations.

(3.2.1) Let $\mathscr{B}(X)$ denote the family of *Borel sets* of X.[7] A measure on $\mathscr{B}(X)$ will be called a *Borel measure* on X.

(3.2.2) Let $\mathfrak{C}(X)$ denote the set of complex-valued continuous functions on X. This is a complex Banach space under pointwise addition and scalar multiplication, with the sup-norm $||f||_u := \sup\{||f(x)|| \mid x \in X\}$.[8]

(3.2.3) Let $\mathfrak{C}_0(X) \subset \mathfrak{C}(X)$ denote the set of complex-valued functions f on X such that for any $\varepsilon > 0$, there exists a compact subspace $K \subset X$ such that $||f(x)|| < \varepsilon$ for any $x \in \complement_X K$. This too is a Banach space.

(3.2.4) Denote by $\mathfrak{C}_{00}(X) \subset \mathfrak{C}_0(X)$ the set of complex-valued functions *with compact support*, i.e., those functions f such that there exists a compact subspace $K \subset X$ such that $f(x) = 0$ for any $x \in \complement_X K$. This is a normed vector space, dense in $\mathfrak{C}_{00}(X)$.

(3.2.5) Denote by $\mathfrak{C}^+_{00}(X) \subset \mathfrak{C}_{00}(X)$ the set of *real*-valued functions f on X such that $f(x) \geq 0$ for any $x \in X$.

(3.2.6) For any Borel measure μ on X and any number $1 \leq p \leq \infty$, we have the Banach space $\mathfrak{L}_p(X, \mu)$ of a.e.-equivalence classes of measurable complex-valued functions on (X, μ) such that $\int_X ||f||^p \, d\mu < \infty$, when $p < \infty$, or, respectively, of locally a.e.-equivalence classes of measurably complex-valued functions on (X, μ) when $p = \infty$. These are equipped with the usual norms $||\cdot||_p$.

(3.2.7) Let $\mathfrak{M}(X)$ be the set of all complex Borel measures on G obtained by the Riesz–Markov theorem; these are the unique complex measures μ_Φ that correspond to complex linear functionals Φ on $\mathfrak{C}_0(X)^*$ such that

$$\Phi(f) = \int_X f \, d\mu_\Phi.$$

The assignment $\Phi \longmapsto \mu_\Phi$ is an isomorphism of Banach spaces $\mathfrak{C}_0(X)^* \cong \mathfrak{M}(X)$.

(3.2.8) If ι is a fixed measure on X, let $\mathfrak{M}_a(X, \iota) \subset \mathfrak{M}(X)$ be the subset of $\mathfrak{M}(X)$ comprised of all complex measures $\mu \in \mathfrak{M}(X)$ that are absolutely continuous with respect to ι. By the Radon–Nikdoym theorem, for any $\mu \in \mathfrak{M}_a(X, \iota)$, there is an a.e.-unique complex-valued function $w \in \mathfrak{L}(X, \iota)$ such that

$$d\mu = u \, d\lambda.$$

This defines an isomorphism of Banach spaces $\mathfrak{M}_a(X, \iota) \cong \mathfrak{L}(X, \iota)$.

Construction of Haar measure. — It is a remarkable fact that in any locally compact group, there is a notion of volume that is *translation invariant*; that is, if you move a set around using elements of the group, the volume of the set does not change. This is the beginning of harmonic analysis.

Definition 3.3. — Suppose G a topological group. A Borel measure μ on G is said to be a *left* (repectively, *right*) *Haar measure* if the following conditions are satisfied.

(3.3.1) For any compact subset $K \subset G$, $\mu(K)$ is finite.

(3.3.2) There exists an open set $U \subset G$ for which $\mu(U) > 0$.

(3.3.3) For any Borel set V and any $g \in G$, the equality $\mu(gV) = \mu(V)$ (resp., the equality $\mu(Vg) = \mu(V)$) obtains.

(3.3.4) The measure μ is *regular* in the sense that for any open set $U \subset G$,

$$\mu(U) = \sup\{\mu(K) \mid K \text{ is compact and } K \subset U\},$$

---

[7]The collection of *Borel sets* is here the smallest σ-algebra containing the set $\mathscr{O}p(X)$ of open sets of X. Other authors (e.g., Halmos) use a slightly different convention.

[8]Note that the resulting topology on $\mathfrak{C}(X)$ is the compact-open topology!

and for any Borel set V,

$$\mu(V) = \inf\{\mu(U) \mid U \text{ is open and } V \subset U\}.$$

The set of all left (resp., right) Haar measures on G will be denoted $\mathscr{H}^{\ell}(G)$ (resp., $\mathscr{H}^{r}(G)$).

If G is abelian, then left Haar measures are right Haar measures and vice versa; consequently, we may in this case simply refer to these as *Haar measures*, and the set of all Haar measures on G will be denoted $\mathscr{H}(G)$.

Exercise 3.4. — For any topological group G, define a map $\mathbf{R}_{>0} \times \mathscr{H}^{\ell}(G) \longrightarrow \mathscr{H}^{\ell}(G)$, $(c,\mu) \longmapsto c\mu$. Show that this defines an action of the group $\mathbf{R}_{>0}$ (2.5.4) on $\mathscr{H}^{\ell}(G)$. The analogous result obviously holds for the set $\mathscr{H}^{r}(X)$ of right Haar measures.

***Theorem 3.5* (Existence and unicity of Haar measure).** — *Suppose G a locally compact group. Then the set $\mathscr{H}^{\ell}(G)$ (respectively, $\mathscr{H}^{r}(G)$) is nonempty, and with the action above, $\mathscr{H}^{\ell}(G)$ (resp., $\mathscr{H}^{r}(G)$) is a torsor under* $\mathbf{R}_{>0}$.[9]

Proof. — The proof employs the Riesz representation theorem; hence our Haar measures will correspond to certain linear functionals on $\mathfrak{C}_{00}(X)$. We begin with a definition:

Definition 3.6. — Suppose S any set. If f is any complex-valued function on G, then for any element $g \in G$, the *left translate* (respectively, the *right translate*) of f by g is the function ${}_g f$ defined by the formula ${}_g f(h) := f(gh)$ (resp., the function f_g defined by the formula $f_g(h) := f(hg)$). The *inversion* of f is the function $f^{\star}$ defined by the formula $f^{\star}(h) := f(h^{-1})$.

Suppose now $\mathfrak{F}$ any set of complex-valued functions on G with the property that if $f \in \mathfrak{F}$, then for any element $g \in G$, one has ${}_g f \in \mathfrak{F}$ (resp., one has $f_g \in \mathfrak{F}$). Suppose $I : \mathfrak{F} \longrightarrow T$ a map. Then I is said to be *left invariant* (resp., *right invariant*) if for any $f \in \mathfrak{F}$ and $g \in G$, the equality $I({}_g f) = I(f)$ obtains (resp., the equality $I(f_g) = I(f)$ obtains). If I is both left and right invariant, then it is said to be *bi-invariant*. One says that I is *inversion invariant* if for any $f \in \mathfrak{F}$, the equality $I(f^{\star}) = I(f)$ obtains.

Our first task is thus to construct a left (resp., right) invariant linear functional on $\mathfrak{C}_{00}(G)$. We restrict ourselves to the left invariant case; the right invariant case is in no way different. It now suffices (by using standard results on extending functionals) to construct a functional I on $\mathfrak{C}_{00}^{+}(G)$ with the following properties.

(3.5.1) For any nonzero $f \in \mathfrak{C}_{00}(G)$, the complex number $I(f)$ is real and positive.
(3.5.2) For any $f, g \in \mathfrak{C}_{00}(G)$, one has $I(f+g) = I(f) + I(g)$.
(3.5.3) For any $\alpha \geq 0$ and any $f \in \mathfrak{C}_{00}(G)$, one has $I(\alpha f) = \alpha I(f)$.
(3.5.4) For any $g \in G$ and $f \in \mathfrak{C}_{00}(G)$, the equality $I({}_g f) = I(f)$ obtains.

Definition 3.7. — A functional I on $\mathfrak{C}_{00}^{+}(G)$ satisfying (3.5.1-4) above will be called a *Haar integral*.

The existence of a Haar integral will suffice show that $\mathscr{H}^{\ell}(G)$ is nonempty; by the Riesz representation theorem, μ will be the unique Borel measure on G such that

$$I(f) = \int_G f\, d\mu$$

for any $f \in \mathfrak{C}_{00}^{+}(G)$. To confirm that $\mathscr{H}^{\ell}(G)$ is a torsor under $\mathbf{R}_{>0}$, it is enough to show that for any Haar measure J, there is a positive real number c such that $I = cJ$. (Why does this suffice?)

Suppose $f_1, f_2 \in \mathfrak{C}_{00}^{+}(G)$; assume f_2 is nonzero. Write $\mathbf{R}_{>0}^{(\infty)}$ for the restricted product $\prod_{j=1}^{\infty} \mathbf{R}_{>0}$, and write $G^{(\infty)}$ for the restricted product $\prod_{j=1}^{\infty} G$. Now set

$$(f_1 : f_2) := \inf\left\{ \sum_{j=1}^{\infty} c_j \mid (c_j)_{j=1}^{\infty} \in \mathbf{R}_{>0}^{(\infty)} \text{ and there exists } (s_j)_{j=1}^{\infty} \in G^{(\infty)} \text{ such that for any } g \in G,\ f_1(x) \leq \sum_{j=1}^{\infty} c_j f_2(s_j g) \right\}.$$

[9] A *torsor* under a group H is a set S with an action $a : H \longrightarrow \operatorname{Aut} S$ such that the *shear map* $H \times S \longrightarrow S \times S$, $(h,s) \longmapsto (a(h,s), s)$ is a bijection.

Exercise 3.8. — Use the fact that f_1 is compactly supported to deduce that $(f_1 : f_2)$ is finite. Moreover, verify the following, for any $f_1, f_2, f_3, f_4 \in \mathfrak{C}^+_{00}(G)$ with f_3 and f_4 nonzero, any $g \in G$, and any $\alpha \geq 0$:

$$\begin{aligned} ({}_g f_1 : f_3) &= (f_1 : {}_g f_3) = (f_1 : f_3); \\ (\alpha f_1 : f_3) &= \alpha(f_1 : f_3); \\ (f_1 + f_2 : f_3) &\leq (f_1 : f_3) + (f_2 : f_3); \\ (f_1 : f_4) &\leq (f_1 : f_3)(f_3 : f_4). \end{aligned}$$

Now we fix a nonzero function $f_0 \in \mathfrak{C}^+_{00}(G)$. For any nonzero function $\phi \in \mathfrak{C}^+_{00}(G)$, and any function $f \in \mathfrak{C}^+_{00}(G)$,

$$I_\phi(f) := \frac{(f : \phi)}{(f_0 : \phi)};$$

This defines a map $I_\phi : \mathfrak{C}^+_{00}(G) \longrightarrow \mathbf{R}$. It is easy to see that $I_\phi(0) = 0$, and for any nonzero $f \in \mathfrak{C}^+_{00}(G)$, one checks that $I_\phi(f) \in [(f_0 : f)^{-1}, (f : f_0)]$. The following facts for any $f, f_1, f_2 \in \mathfrak{C}^+_{00}(G)$ with $||f_1||_u \leq ||f_2||_u$, any $g \in G$, and any $\alpha > 0$, are immediate consequences of the previous exercise:

$$\begin{aligned} I_\phi({}_g f) &= I_\phi(f); \\ I_\phi(\alpha f) &= \alpha I_\phi(f); \\ I_\phi(f_1 + f_2) &\leq I_\phi(f_1) + I_\phi(f_2); \\ I_\phi(f_1) &\leq I_\phi(f_2). \end{aligned}$$

Now contemplate the product

$$X := \prod_{f \in \mathfrak{C}^+_{00}(G),\ f \neq 0} \left[\frac{1}{(f_0 : f)}, (f : f_0) \right].$$

By the Tychonoff product theorem, X is compact. For any nonzero function $\phi \in \mathfrak{C}^+_{00}(G)$, the function I_ϕ can be viewed as a point whose f-th component is $I_\phi(f)$ for any nonzero $f \in \mathfrak{C}^+_{00}(G)$. Now let $\mathscr{U}$ be the set of all neighborhoods of e in G. For each $U \in \mathscr{U}$, choose a function $\phi_U \in \mathfrak{C}^+_{00}(G)$ such that for any $g \in \complement_G U$, $\phi_U(g) = 0$. Now $\mathscr{U}$ can be given a partial order by reverse inclusion, so that $U \leq V$ if and only if $V \supset U$, so the assignment $U \longmapsto I_{\phi_U}$ is a net in X. Since X is compact, there is a subnet that converges to an element $I \in X$, which can be regarded as a map $I : \mathfrak{C}^+_{00}(G) \longrightarrow \mathbf{R}_{\geq 0}$ (sending 0 to 0).

It is easy to see that I is real and positive on nonzero functions; I also automatically left-invariant. So it remains to show that I satisfies (3.5.2-3). This follows from the following (challenging) exercise.

Exercise 3.9. — Suppose $f_1, f_2, \ldots, f_n \in \mathfrak{C}^+_{00}(G)$ be nonzero functions, and suppose δ, ε two positive real numbers. Then there is a neighborhood U of e in G such that for any nonzero $\phi \in \mathfrak{C}^+_{00}(G)$ with the property that for any $g \in \complement_G U$, $\phi(g) = 0$, and any finite set $\{\lambda_j\}_{j=1}^m$ of elements of $[0, \varepsilon]$, the inequality

$$\sum_{j=1}^m \lambda_j I_\phi(f_j) \leq I_\phi\left(\sum_{j=1}^m \lambda_j f_j\right) + \delta.$$

obtains.

Exercise 3.10. — Show that for any functional J on $\mathfrak{C}^+_{00}(G)$ satisfying (3.5.1-4), then for any nonzero functions $f_1, f_2 \in \mathfrak{C}^+_{00}(G)$, the equality

$$\frac{J(f_1)}{I(f_1)} = \frac{J(f_2)}{I(f_2)}$$

obtains.

This completes the proof. □

Exercise 3.11. — Show that if G is discrete, then a (left or right) Haar measure on G is a constant multiple of the counting measure. For G discrete, the counting measure will be called the *conormalized* Haar measure.(10)

Exercise 3.12. — Show that if G is a nondiscrete locally compact group, then for any Haar measure λ on G, the measure $\lambda(A)$ of any countable subset $A \subset G$ is zero.

Exercise 3.13. — Suppose $\mathscr{G} = \{G_\alpha\}_{\alpha\in\Lambda}$ a family of locally compact groups, and suppose, for every $\alpha \in \Lambda$, that λ_α is a Haar measure on G_α. Show the product measure on $\prod_{\alpha\in\Lambda} G_\alpha$ is a Haar measure, and show that there is a unique Haar measure on the restricted product $\coprod_{\alpha\in\Lambda} G_\alpha$ whose restriction to $\prod_{\alpha\in K} G_\alpha$ is the product measure for every finite subset $K \subset \Lambda$.

Exercise 3.14. — Suppose G a compact group. Then show that there is a unique left Haar measure λ on G such that $\lambda(G) = 1$. This will be called the *normalized* Haar measure for G.

Exercise 3.15. — Suppose G a locally compact group, suppose λ a left Haar measure on G, and suppose H a topological group that is either σ-comopact or else contains a countable dense subset. Suppose $\rho : G \longrightarrow H$ a homomorphism such that there exists a λ-measurable set $A \subset G$ of finite, nonzero measure for which the set $\rho^{-1}(U \cap \rho(G)) \cap A$ is λ-measurable for any open set $U \subset H$. Then show that ρ is in fact continuous.

Deduce that any homomorphism $\rho : G \longrightarrow \mathbf{GL}_n(\mathbf{C}),\ g \longmapsto (a_{ij}(g))_{i,j}$, for which there is a λ-measurable set $A \subset G$ of finite, nonzero measure such that each function $g \longmapsto a_{ij}(g)$ is measurable on A, is continuous.

Exercise 3.16. — Suppose G a locally compact group, and let μ be a left Haar measure on G. For any nonzero function $f \in \mathfrak{C}^+_{00}(G)$, and for any $g \in G$, set

$$\Delta(g) := \frac{\int_G f_{g^{-1}}\, d\mu}{\int_G f\, d\mu}.$$

Show that $\Delta(x)$ is independent of the choice of μ or f. Moreover, check that $\Delta(g)$ is positive for any $g \in G$, and that $\Delta : G \longrightarrow \mathbf{R}_{>0}$ is continuous. Finally, show that Δ is a group homomorphism. This continuous homomorphism is called the *modular function* of G. If Δ is the constant function at 1, then G is said to be *unimodular*. A group G is thus unimodular if and only if every left Haar measure is a right Haar measure. Of course any LCA group is unimodular.

Exercise 3.17. — Show that any compact group is unimodular. [Hint: what can you say about the image of Δ?]

Exercise 3.18. — For any subset A of G, any $g \in G$, and any left Haar measure μ on G, show that

$$\mu(Ax) = \Delta(x)\mu(A).$$

Conclude that if f is a nonnegative μ-measurable function on G, then for any $g \in G$,

$$\int_G f_g\, d\mu = \frac{1}{\Delta(g)} \int_G f\, d\mu.$$

Definition 3.19. — Suppose now G a locally compact group, and suppose H a closed subgroup of G. A function ψ on the quotient set G/H can be left translated by an element $g \in G$, yielding a function ${}_g\psi$ on G/H defined by ${}_g\psi(kH) = \psi(gkH)$. A nonzero functional J on $\mathfrak{C}_{00}(G/H)$ is said to be *relatively invariant* if the following conditions are satisfied.

(3.19.1) For any $\psi \in \mathfrak{C}^+_{00}(G/H)$, one has $J(\psi) \in [0,\infty)$.
(3.19.2) For any $\psi_1, \psi_2 \in \mathfrak{C}_{00}(G/H)$, one has $J(\psi_1 + \psi_2) = J(\psi_1) + J(\psi_2)$.
(3.19.3) For any $\alpha \in \mathbf{C}$ and $\psi \in \mathfrak{C}_{00}(G/H)$, one has $J(\alpha\psi) = \alpha J(\psi)$.
(3.19.4) For any $g \in G$ there is a number $\chi_J(g)$ such that for any $\psi \in \mathfrak{C}_{00}(G/H)$, the equality $I({}_g\psi) = \chi_J(g) I(\psi)$ obtains.

(10)The logic behind this slightly odd terminology will be explained in due course.

Exercise 3.20. — Suppose G a locally compact group, and suppose H a closed subgroup of G. Suppose J a relatively invariant functional on $\mathfrak{C}_{00}(G/H)$. Then the associated function χ_J on G is real, positive, and continuous. [Hint: for continuity, show that χ is continuous at e by verifying the following Lemma: Suppose ψ a continuous, complex-valued function on G/H such that for any $\delta > 0$ there is a compact subset F of G outside of whose image in G/H one has $\|\psi\| < \delta$. Then for any $\varepsilon > 0$, there is a symmetric neighborhood V of $e \in G$ such that for any $g, k \in G$ with $kg^{-1} \in V$, one has $\|\psi(gH) - \psi(kH)\| < \varepsilon$.]

Exercise 3.21. — Suppose G a locally compact group, and suppose H a closed subgroup of G. Suppose I a left Haar integral on H. Show the map $\Psi_I : \mathfrak{C}_{00}(G) \longrightarrow \mathfrak{C}_{00}(G/H)$ defined by

$$(\Psi_I f)(gH) := I({}_g f)$$

is a surjective linear map. Deduce that if J is a relatively invariant linear functional on $\mathfrak{C}_{00}(G/H)$, then for any $h \in H$,

$$\chi_J(h) = \frac{\Delta_G(h)}{\Delta_H(h)},$$

where Δ_G is the modular function of G, and Δ_H is the modular function of H. [Hint: show the functional $K(f) := J(\Psi_I(\chi_J f))$ is a Haar integral for G.]

Exercise 3.22. — Suppose G a locally compact group, and suppose H a closed *normal* subgroup of G. Show that the Haar integral on G/H defines a relatively invariant linear functional on $\mathfrak{C}_{00}(G/H)$, so that $\chi_I = 1$. Conclude that $\Delta_G = \Delta_H$ in this case.

Exercise 3.23. — Suppose G a locally compact group, and suppose H a closed subgroup of G. Suppose, moreover, that there is an extension of the homomorphism $\Delta_G/\Delta_H : H \longrightarrow \mathbf{R}_{>0}$ to a continuous homomorphism $\chi : G \longrightarrow \mathbf{R}_{>0}$. Then show that there is a relatively invariant functional J on $\mathfrak{C}_{00}(G/H)$ such that $\chi_J = \chi$. [Hint: if $q : G \longrightarrow G/H$ is the quotient map, show that $J(f) = I(\chi^{-1}(q \circ f))$ is desired functional.]

Convolutions. — On any locally compact group G, we may contemplate the space of continuous functions that vanish at ∞. The linear dual to this space carries an important algebra structure, called *convolution*, which reflects certain properties of G itself.

Definition 3.24. — Suppose G a group, and suppose $\mathfrak{F}$ a vector space of complex-valued functions on G. Suppose $\mathfrak{F}$ is closed under left translations in the sense that ${}_g f \in \mathfrak{F}$ whenever $f \in \mathfrak{F}$ and $g \in G$. If $\Psi \in \mathfrak{F}^*$ is a complex-valued linear functional on $\mathfrak{F}$, then for any function $f \in \mathfrak{F}$, write $\overline{\Psi} f$ for the complex-valued function on G defined by the formula $\overline{\Psi} f(g) := M({}_g f)$. If $\overline{\Psi} f \in \mathfrak{F}$, then for any linear functional $\Phi \in \mathfrak{F}^*$, there is a linear functional $\Phi * \Psi \in \mathfrak{F}^*$ such that $(\Phi * \Psi) f := \Phi(\overline{\Psi} f)$.

Put differently, our assumption that $\mathscr{F}$ is closed under left translations is the claim that the assignment $(g, f) \longmapsto {}_g f$ defines a map $G \times \mathfrak{F} \longrightarrow \mathfrak{F}$, which can be reinterpreted as a map $\mathfrak{F} \longrightarrow \mathrm{Map}(G, \mathfrak{F})$; the assignment $f \longmapsto \overline{\Psi} f$ is the composite

$$\mathfrak{F} \longrightarrow \mathrm{Map}(G, \mathfrak{F}) \xrightarrow{\Psi \circ -} \mathrm{Map}(G, \mathbf{C}).$$

If $\overline{\Psi} f \in \mathfrak{F}$ for every $f \in \mathfrak{F}$, then this map factors through $\mathfrak{F} \subset \mathrm{Map}(G, \mathbf{C})$, whence we have a map $\mathfrak{F} \longrightarrow \mathfrak{F}$. Now if $\Phi \in \mathfrak{F}^*$ is a linear functional, then composition with yields a new linear functional $\Phi \star \Psi$. This new linear functional is called the *convolution* of Φ and Ψ.

A vector subspace $\mathfrak{E} \subset \mathfrak{F}^*$ is called a *convolution algebra* if for any $\Phi, \Psi \in \mathfrak{E}$, the convolution $\Phi * \Psi$ exists and is an element of $\mathfrak{E}$.

3.25. — Suppose, for the remainder of this section, that G is a locally compact group, and λ is a chosen left Haar measure on G.

Exercise 3.26. — Show that for any function $f \in \mathfrak{C}_0(G)$ and any linear functional $\Phi \in \mathfrak{C}_0(G)^*$, the function $\overline{\Phi} f$ is in $\mathfrak{C}_0(G)^*$. Conclude that $\mathfrak{C}_0(G)^*$ is a convolution algebra, and in fact it is a Banach algebra.

Exercise 3.27. — Consider, for any element $g \in G$, the functional $E_g \in \mathfrak{C}_0(G)^*$ defined by $E_g(f) = f(g)$; i.e., E_g is evaluation at g. Show that for any elements $g, h \in G$, the identity $E_g * E_h = E_{gh}$ obtains. Moreover, E_e is the unit for $- * -$.

Exercise 3.28. — By the Riesz-Markov theorem, on the Banach space $\mathfrak{M}(G)$ there is a convolution product on $- * -$, giving $\mathfrak{M}(G)$ the structure of a Banach algebra. Show that if $\Phi, \Psi \in \mathfrak{C}_0(G)^*$, then

$$(\Phi * \Psi)(f) = \int_G f \, d(\mu_\Phi * \mu_\Psi) = \int_G \int_G f(gh) \, d\mu_\Phi(g) \, d\mu_\Psi(h).$$

Conclude that for any $\mu, \nu \in \mathfrak{M}(G)$, and for any $||\mu * \nu||$-measurable subset $A \subset X$,

$$(\mu * \nu)(A) = (\mu \times \nu)(\tau^{-1}A) = \int_G \nu(g^{-1}A) \, d\mu(g) = \int_G \mu(Ah^{-1}) \, d\nu(h),$$

where $\tau : G \times G \longrightarrow G$ is the group multiplication.

Exercise 3.29. — Suppose G a locally compact group, and suppose H a closed normal subgroup of G. Suppose λ_G and λ_H fixed Haar measures on G and H, respectively. Show that a Haar measure $\lambda_{G/H}$ can be chosen so that

$$\int_G f(x) \, d\lambda_G(x) = \int_{G/H} \int_H f(x\xi) \, d\lambda_H(\xi) \, d\lambda_G(xH).$$

This is sometimes known as *Weil's identity*.

Exercise 3.30. — Show that if A is an LCA group, use Fubini to show that the Banach algebra $\mathfrak{M}(A)$ is commutative.

Exercise 3.31. — Show that if $\mu \in \mathfrak{M}(G)$ and $\nu \in \mathfrak{M}_a(G, \lambda)$, then both $\mu * \nu$ and $\nu * \mu$ are also absolutely continuous with respect to λ. Hence $\mathfrak{M}_a(G, \lambda)$ is a two-sided Banach ideal in $\mathfrak{M}(G)$.

Deduce that for any $\mu \in \mathfrak{M}(G)$ and any $f \in \mathfrak{L}_1(G, \lambda)$, there are convolutions $\mu * f, f * \mu \in \mathfrak{L}_1(G, \lambda)$ such that

$$(\mu * f)(g) = \int_G f(h^{-1}g) \, d\mu(h) \qquad \text{and} \qquad (f * \mu)(g) = \int_G \frac{1}{\Delta(h)} f(gh^{-1}) \, d\mu(h)$$

for almost all $g \in G$. In particular, for any $f_1, f_2 \in \mathfrak{L}_1(G, \lambda)$, there is a convolution $f_1 * f_2 \in \mathfrak{L}_1(G, \lambda)$ such that

$$\begin{aligned} (f_1 * f_2)(g) &= \int_G f_1(h) f_2(h^{-1}g) \, d\lambda(h) = \int_G f_1(gh) f_2(h^{-1}) \, d\lambda(h) \\ &= \int_G \frac{1}{\Delta(h)} f_1(gh^{-1}) f_2(h) \, d\lambda(h) = \int_G \frac{1}{\Delta(h)} f_1(h^{-1}) f_2(hg) \, d\lambda(h). \end{aligned}$$

Exercise 3.32. — Suppose now $1 \leq p \leq \infty$. Show that for any Borel measurable function $f \in \mathfrak{L}_p(G, \lambda)$ and for any complex measure $\mu \in \mathfrak{M}(G)$, there is a subset $Z \subset G$ that is λ-null if $p < \infty$ and locally λ-null if $p = \infty$ such that for any $g \in \complement_G Z$, the integral

$$(\mu * f)(g) = \int_G f(h^{-1}g) \, d\mu(h)$$

exists and is finite. Defining $\mu * f$ as 0 where it is not defined, show that $\mu * f \in \mathfrak{L}_p(G, \lambda)$, with

$$||\mu * f||_p \leq ||\mu|| \cdot ||f||_p.$$

Hence $\mathfrak{L}_p(G, \lambda)$ is a left Banach module over the Banach algebra $\mathfrak{M}(G)$, and consequently over $\mathfrak{L}_1(G, \lambda)$ as well.

Examples of Haar measure. — Let us look at a few examples of Haar measure. These examples get pretty complicated pretty quickly (evidence for the power of the theory!), so we will stick with simple examples for now.

Exercise 3.33. — Consider the additive group $\mathbf{R}$. Show that the ordinary Lebesgue measure on $\mathbf{R}$ is a Haar measure. More generally, the Lebesgue measure is a Haar measure on any $\mathbf{R}^n$.

Given a vector group E (i.e., a finite-dimensional real vector space), the choice of a basis specifies an isomorphism $E \cong \mathbf{R}^n$, under which we can transport the Haar measure. The group $\mathbf{GL}(E)$ therefore acts transitively on $\mathscr{H}(E)$. However, this action is not faithful: since the action $\mathbf{GL}(E) \times \mathscr{H}(E) \longrightarrow \mathscr{H}(E)$ is $(\phi, \mu) \longmapsto \|\det \phi\| \mu$, the action factors through $\mathbf{GL}(E)/(\mathbf{SL}(E) \times \mathbf{Z}/2) \cong \mathbf{R}$.

Example 3.34. — Consider the circle $\mathbf{U}(1)$. This is a compact group, so there is a unique left Haar measure λ on $\mathbf{U}(1)$ such that $\lambda(\mathbf{U}(1)) = 1$. For any $f \in \mathfrak{C}^+_{00}(\mathbf{U}(1))$, one may interpret f as a function on the interval $[0, 1]$ so that

$$I(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{2\pi\theta\sqrt{-1}})\, d\mu(\theta)$$

is the normalized Haar integral, where μ is usual Lebesgue measure on $[0, 1]$.

Exercise 3.35. — Suppose n a nonnegative integer. Then we can regard $\mathbf{GL}_n(\mathbf{R})$ as a subspace of $\mathbf{R}^{n^2}$. Verify that the formula

$$\mu(S) = \int_S \frac{1}{\|\det A\|^n}\, d\mu(A),$$

where μ is the usual Lebesgue measure on $\mathbf{R}^{n^2}$.

Exercise 3.36. — Using the Euler angles, show that, for any $f \in \mathfrak{C}^+_{00}(G)$,

$$I(f) = \frac{1}{8\pi^2} \int_0^{\pi} \int_0^{2\pi} \int_0^{2\pi} f(Z(\psi)X(\theta)Z(\phi)) \sin\theta\, d\mu(\psi)\, d\mu(\phi)\, d\mu(\theta)$$

is the normalized Haar integral on $\mathbf{SO}(3)$, where μ is the usual Lebesgue measure.

4. Pontryagin duality

Character groups of locally compact abelian groups. — The set of 1-dimensional representations, or *characters* of a locally compact abelian (LCA) group completely characterize that group. We now discuss the theory of characters of and LCA group now.

Definition 4.1. — Suppose G a locally compact group. Then a *character* of G is a continuous homomorphism

$$\chi : G \longrightarrow \mathbf{U}(1).$$

The *character group*, *(Pontryagin) dual group* of G is the (abelian) group $\widehat{G}$ of characters of G, under the pointwise product, equipped with the compact-open topology.

Exercise 4.2. — Suppose G a locally compact group. Show that the topology on $\widehat{G}$ is generated by sets of the form

$$\{h \in G \mid \text{for any } g \in K, \|gh - \chi(g)\| < \varepsilon\},$$

where $K \subset G$ is compact, $\chi \in \widehat{G}$, and $\varepsilon > 0$.

Exercise 4.3. — For any locally compact group G, show that the map

$$\begin{aligned} \widehat{G} \times G &\longrightarrow \mathbf{U}(1) \\ (\chi, g) &\longmapsto \chi(g) \end{aligned}$$

is continuous.

Exercise 4.4. — Show that $\widehat{G}$ is an LCA group for any group G. Moreover, show that the assignment $G \longmapsto \widehat{G}$ is a *contravariant functor*, in the sense that for any continuous homomorphism $\phi : G' \longrightarrow G$, there is a dual continuous homomorphism

$$\widehat{\phi} : \widehat{G} \longrightarrow \widehat{G}'$$

given by $\widehat{\phi}(\chi) = \chi \circ \phi$.

Exercise 4.5. — Suppose G a locally compact group. Show that the intersection $\bigcap_{\chi \in \widehat{G}} \ker \chi$ is the closure C of the commutator subgroup $[G : G]$ of G.[11] Deduce that the groups $\widehat{G}$ and $\widehat{G/C}$ are topologically isomorphic.

4.6. — It follows from the previous exercise that the character group of a locally compact group only sees the abelian topological quotient. Let us assume from now on that A is an LCA group with chosen Haar measure λ.[12]

Exercise 4.7. — The topological group $\widehat{A}$ is an LCA group. If A is discrete, then $\widehat{A}$ is compact, and if A is compact, then $\widehat{A}$ is discrete. In particular, show that if A is profinite, then $\widehat{A}$ is torsion discrete, and if A is torsion discrete, then $\widehat{A}$ is profinite.

Exercise 4.8. — Suppose A a compact abelian group, and suppose $\Gamma \subset \widehat{A}$ a subgroup such that for every $g \in A$, either $g = e$ or else there is a character $\chi \in \Gamma$ such that $\chi(g) \neq 1$. Then $\Gamma = \widehat{A}$. [Hint: Stone-Weierstraß.]

Exercise 4.9. — Suppose A an LCA group, suppose B a closed subgroup of A, and suppose $\psi \in \widehat{B}$ a character of B. Then if B is either compact or open, there is a character χ of G extending ψ.

Exercise 4.10. — Suppose $\mathscr{A} = \{A_\alpha\}_{\alpha \in \Lambda}$ any family of compact abelian groups. Then show that there is a natural topological isomorphism

$$\coprod_{\alpha \in \Lambda} \widehat{A}_\alpha \cong \widehat{\prod_{\alpha \in \Lambda} A_\alpha}.$$

Show that the same result holds if $\mathscr{A} = \{A_\alpha\}_{\alpha \in \Lambda}$ is a finite family of LCA groups. [Hint: define a homomorphism from the left to the right by the sending any tuple $(\chi_\alpha)_{\alpha \in \Lambda}$ of characters in the restricted product to the function $(g_\alpha)_{\alpha \in \Lambda} \longmapsto \prod_{\alpha \in \Lambda} \chi_\alpha(g_\alpha)$.]

Definition 4.11. — Suppose A an LCA group, and suppose B a closed subgroup. Define the *annihilator* $B^\perp \subset \widehat{A}$ of B in $\widehat{A}$ as the subset of characters $\chi \in \widehat{A}$ such that $\chi(h) = 1$ for any $h \in A$.

Exercise 4.12. — Suppose A an LCA group, and suppose B a closed subgroup; write $q : A \longrightarrow A/B$ for the quotient map. Show that $B^\perp$ is a closed subgroup of $\widehat{A}$, and show that the map $\chi \longmapsto \chi \circ q$ is a topological isomorphism $\widehat{A/B} \cong B^\perp$. Deduce that if $g \in A$ is an element not contained in B, then there exists a character $\chi \in \widehat{A}$ such that $\chi(g) \neq 1$.

Exercise 4.13. — Suppose A an LCA group, and suppose B an open subgroup. Show that the natural continuous homomorphism $\widehat{A} \longrightarrow \widehat{B}$ induced by the inclusion is open. Deduce that $\widehat{B} \cong \widehat{A}/B^\perp$.

Exercise 4.14. — (4.14.1) Show that the character group of any vector group E is isomorphic to the dual vector space E^*. [Hint: For any $\xi \in E^*$, contemplate the map $\chi_\xi : x \longmapsto e^{2\pi\sqrt{-1}\xi(x)}$.]

(4.14.2) Show that for every nonnegative integer m, the Pontryagin dual of $\mathbf{Z}^m$ is the torus $\mathbf{T}^m$.

(4.14.3) Show that, in the other direction, for every nonnegative integer m, the Pontryagin dual of $\mathbf{T}^m$ is $\mathbf{Z}^m$.

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[11] The commutator subgroup $[G : G]$ of a group G is the group generated by elements of the form $ghg^{-1}h^{-1}$. If C is the closure of $[G : G]$ in G, then G/C is the abelian topological quotient, in the sense that for any LCA group A, any continuous homomorphism $G \longrightarrow A$ factors uniquely through the quotient $G \longrightarrow G/C$.

[12] I no longer have to say "left" or "right" Haar measure, because A is a abelian!

(4.14.4) If A is a finite abelian group, show that $\widehat{A}$ is (noncanonically) isomorphic with itself. [Hint: show this first for cyclic groups $\mathbf{Z}/m$ by contemplating the map $k \longmapsto e^{2\pi\sqrt{-1}(k/m)}$; then use the classification of finite abelian groups.]

(4.14.5) Show that the dual group of $\widehat{\mathbf{Z}}$ (the profinite completion of the integers!) is the discrete group $\mathbf{Q}/\mathbf{Z}$. [Hint: if we write $\widehat{\mathbf{Z}}$ as a limit of its finite quotient, then its dual group must contain each of these finite groups as a subgroup.]

(4.14.6) Show that, in the other direction, the dual group of $\mathbf{Q}/\mathbf{Z}$ is $\widehat{\mathbf{Z}}$.

(4.14.7) For any prime number p, show that the dual group of $\mathbf{Z}_p$ is the Prüfer p-group[13] $\mathbf{Z}[\frac{1}{p}]/\mathbf{Z}$, and show that dual group of $\mathbf{Z}[\frac{1}{p}]/\mathbf{Z}$ is $\mathbf{Z}_p$.

Exercise 4.15 **(For students with experience with field theory)**. — Here is a more exotic example. Suppose K a number field, and recall the group of adèles $\mathbf{A}_K$ of **??**; this an LCA group. Show that $\mathbf{A}_K$ is its own Pontryagin dual, and use this to show that the Pontryagin dual of K with the discrete topology is $\mathbf{A}_K$.

The Fourier transform and the inverse Fourier transform. — Fourier analysis is one of the cornerstones of modern mathematics. In this subsection, we generalize the basic constructions — the *Fourier transform* and the *inverse Fourier transform* — to their most natural context: that of LCA groups.

4.16. — Suppose here A an LCA group.

Exercise 4.17. — Suppose $\lambda \in \mathscr{H}(A)$, and suppose Φ a (nonzero) multiplicative linear functional on $\mathfrak{M}_a(A,\lambda)$. Show that there is a unique character χ_Φ of G such that for any $\mu \in \mathfrak{M}_a(A,\lambda)$,

$$\Phi(\mu) = \int_A \overline{\chi_\Phi}\, d\mu.$$

This defines a homeomorphism between the set of multiplicative linear functionals on $\mathfrak{M}_a(A,\lambda)$ (with the subspace topology inherited from $\mathfrak{M}_a(A,\lambda)^*$) and the character group $\widehat{A}$.

Exercise 4.18. — Suppose now A compact, and suppose $\lambda \in \mathscr{H}(A)$. Verify that for any $\chi \in \widehat{A}$,

$$\int_A \chi\, d\lambda = \delta_1(\chi)\lambda(A),$$

where δ_1 is the Kronecker delta function at 1 on the discrete group $\widehat{A}$.

Definition 4.19. — For any $\mu \in \mathfrak{M}(A)$, denote by $\hat{\mu}$ the complex-valued function on $\widehat{A}$ such that

$$\hat{\mu}(\chi) := \int_A \overline{\chi(g)}\, d\mu(g).$$

This is called the *Fourier–Stieltjes transform* of μ. If now $\mu \in \mathfrak{M}_a(A,\lambda)$ for a Haar measure $\lambda \in \mathscr{H}(A)$, so that $d\mu = f\ d\lambda$ for some $f \in \mathfrak{L}_1(A,\lambda)$, then we may write $\hat{f}$ for $\hat{\mu}$, and we call $\hat{f}$ the *Fourier transform of f relative to λ.*

Dually, for any $\mu \in \mathfrak{M}(\widehat{A})$, denote by $\check{\mu}$ the complex-valued function on A such that

$$\check{\mu}(g) := \int_{\widehat{A}} \chi(g)\, d\mu(\chi).$$

This is called the *inverse Fourier–Stieltjes transform* of μ. If now $\mu \in \mathfrak{M}_a(\widehat{A},\lambda)$ for a Haar measure $\lambda \in \mathscr{H}(\widehat{A})$, so that $d\mu = f\ d\lambda$ for some $f \in \mathfrak{L}_1(\widehat{A},\lambda)$, then we may write $\hat{f}$ for $\hat{\mu}$, and we call $\hat{f}$ the *inverse Fourier transform of f relative to λ.*

Exercise 4.20. — Confirm the following properties about the Fourier–Stieltjes transform (respectively, about the inverse Fourier–Stieltjes transform) for any $\mu,\nu \in \mathfrak{M}(A)$ (resp., for any $\mu,\nu \in \mathfrak{M}(\widehat{A})$) and any $\alpha \in \mathbf{C}$:

(4.20.1) $\widehat{\mu+\nu} = \hat{\mu} + \hat{\nu}$ (resp., $(\mu+\nu)^\vee = \check{\mu} + \check{\nu}$);

[13] The Prüfer p-group is the subgroup of the circle consisting of the set of all p^n-th roots of unity for $n > 0$.

(4.20.2) $\widehat{\alpha\mu} = \alpha\hat{\mu}$ (resp., $(\alpha\mu)^\vee = \alpha\check{\mu}$);
(4.20.3) $\widehat{\mu * \nu} = \hat{\mu} * \hat{\nu}$ (resp., $(\mu * \nu)^\vee = \check{\mu} * \check{\nu}$);
(4.20.4) $\sup\{\|\hat{\mu}(\chi)\| \mid \chi \in \widehat{A}\} \leq \|\mu\|$ (resp., $\sup\{\|\check{\mu}(g)\| \mid g \in A\} \leq \|\mu\|$).
(4.20.5) If $\mu \neq 0$, then $\hat{\mu} \neq 0$ (resp., $\check{\mu} \neq 0$).

Exercise 4.21. — Show that if $f \in \mathfrak{L}_1(A, \lambda)$ (respectively, if $f \in \mathfrak{L}_1(\widehat{A}, \lambda)$) for a Haar measure $\lambda \in \mathscr{H}(A)$ (resp., a Haar measure $\lambda \in \mathscr{H}(\widehat{A})$), then $\hat{f} \in \mathfrak{C}_0(\widehat{A})$ (resp., $\check{f} \in \mathfrak{C}_0(A)$). [Hint: show that for any $\varepsilon > 0$, there is a neighborhood U of e in G such that $\|_g f - f\|_1 < \varepsilon$ for any $g \in U$.]

Definition 4.22. — For any measure $\mu \in \mathfrak{M}(A)$, denote by $\mathfrak{T}(A)$ the vector space of complex-valued functions on A spanned by the characters $\chi \in \widehat{A}$. The elements of $\mathfrak{T}(A)$ will be called *trigonometric polynomials*.

Exercise 4.23. — Show that for any measure $\mu \in \mathfrak{M}(A)$ and for any $1 \leq p < \infty$, the vector space $\mathfrak{T}(A)$ is a dense subspace of $\mathfrak{L}_p(A, \|\mu\|)$. [Hint: Stone–Weierstraß.]

Exercise 4.24. — Use the previous exercise to show that the Fourier–Stieltjes transform $\mu \longmapsto \hat{\mu}$ (respectively, the inverse Fourier–Stieltjes transform $\mu \longmapsto \check{\mu}$) is an isomorphism

$$\mathfrak{M}(A) \longrightarrow \mathfrak{C}_u(\widehat{A}) \qquad \left(\text{resp.,} \qquad \mathfrak{M}(\widehat{A}) \longrightarrow \mathfrak{C}_u(A) \quad \right)$$

that restricts to an isomorphism from $\mathfrak{L}_1(A)$ (resp., from $\mathfrak{L}_1(\widehat{A})$) onto a dense subalgebra of $\mathfrak{C}_0(\widehat{A})$ (resp., a dense subalgebra of $\mathfrak{C}_0(A)$). This subalgebra is known as the *Wiener algebra* $\mathfrak{A}(\widehat{A})$ (resp., $\mathfrak{A}(A)$).

Exercise 4.25. — Recall that if E is a vector group, then its dual group is the dual vector space E^*; a vector ξ can be regarded as a character $x \longmapsto e^{2\pi\sqrt{-1}\xi(x)}$. Now for any Haar measure $\lambda \in \mathscr{H}(E)$, the Fourier transform of any function $f \in \mathfrak{L}_1(E, \lambda)$ may be written as

$$\hat{f}(\xi) = \int_E f(x) e^{-2\pi\sqrt{-1}\xi(x)}\, d\lambda(x).$$

Similarly, the inverse Fourier transform of any function $f \in \mathfrak{L}_1(E^*, \lambda)$ may be written as

$$\check{f}(x) = \int_{E^*} f(\xi) e^{2\pi\sqrt{-1}\xi(x)}\, d\lambda(\xi).$$

Exercise 4.26. — The dual group to the torus $\mathbf{T}^n$ is $\mathbf{Z}^n$. Let us take the normalized Haar measure λ on $\mathbf{T}^n$ and the conormalized (counting) Haar measure on $\mathbf{Z}^n$. Show that for any sequence $a = (a_m)_{m \in \mathbf{Z}^n} \in \ell_1(\mathbf{Z}^n)$ of complex numbers, the function

$$\check{a}(\theta_1, \theta_2, \ldots, \theta_n) = \frac{1}{(2\pi)^n} \sum_{m \in \mathbf{Z}^n} a_m e^{2\pi\sqrt{-1}(m_1\theta_1 + \cdots + m_n\theta_n)}$$

is a continuous function on $\mathbf{T}^n$.

Pontryagin duality. —

Exercise 4.27. — Suppose A an LCA group. Denote by A' the *double dual* $\widehat{\widehat{A}}$ of A. For any $g \in A$, let $\eta_A(g)$ be the function on $\widehat{A}$ defined by the formula $\eta_A(g)(\chi) := \chi(g)$. Show that this defines a continuous homomorphism

$$\eta_A : A \longrightarrow A'$$

Theorem 4.28. — *For any LCA group A, the map η_A is a topological isomorphism.*

Proof. — The proof proceeds in stages.

Exercise 4.29. — Use 4.8 to show that the theorem holds for any compact group and any discrete group.

Exercise 4.30. — Use the structure theorem 2.73 of compactly generated LCA groups to show that the theorem holds for any compactly generated LCA group.

Exercise 4.31. — Use 2.63 to complete the proof. □

4.32. — It would be nice to have a proof that doesn't rely on classification results, but I do not know of one.

Exercise 4.33 **(For students who have experience with elementary category theory)**
Show that the category **LCA** of LCA groups is equivalent to its opposite. Moreover, since any finite-dimensional real vector space is also an LCA group, show that the equivalence $\mathbf{LCA} \simeq \mathbf{LCA}^{\mathrm{op}}$ restricts to the equivalence $\mathbf{Vect}_{\mathbf{R}} \simeq \mathbf{Vect}_{\mathbf{R}}^{\mathrm{op}}$ given by taking the dual vector space.

Fourier inversion, Plancherel theorem, and Parseval's identity. — Now we quickly extract three seminal results in harmonic analysis. We are forced to begin by quoting a key technical lemma.

4.34. — Suppose A an LCA group.

Lemma 4.35. — *Suppose $A \cong A' \times E$, where A' is an LCA group with an open compact subgroup $K \subset A'$, and E is a vector group of real dimension d. Suppose $\Delta \subset \widehat{A}$ a σ-compact subgroup of the form $E^* \times \Delta_0$, where Δ_0 is a σ-compact subgroup containing $K^{\perp}$. Then there are a sequence $\{w_n\}_{n=1}^{\infty}$ of functions in $\mathfrak{C}_0^+(\widehat{A}) \cap \mathfrak{L}_1(\widehat{A})$ and a sequence $\{\phi_n\}_{n=1}^{\infty}$ in $\mathfrak{C}_0^+(A) \cap \mathfrak{L}_1(A)$ with the following properties:*

(4.35.1) the image of w_n is contained in $[0,1]$ for any n;
(4.35.2) $w_n \leq \chi_\Delta$, where χ_Δ denotes the characteristic function of Δ, for any n;
(4.35.3) $w_n = w_n^\star$ [3.6] for any n;
(4.35.4) $\lim_{n\to\infty} w_n = \chi_\Delta$;
(4.35.5) $\check{w}_n = \phi_n$ for any n;
(4.35.6) $\int_G \phi_n \, d\lambda$ for any n;
(4.35.7) $\phi_n = \phi_n^\star$ for any n;
(4.35.8) if $\psi \in \Delta$, then $\lim_{n\to\infty} \psi * \phi_n = \psi$;
(4.35.9) if $\psi \in \complement_{\widehat{A}}\Delta$, then $\psi * \phi_n = 0$ for any n;
(4.35.10) if $f \in \mathfrak{C}_{00}(A)$ and $\{\psi \in \widehat{A} \mid \hat{f}(\psi) \neq 0\} \subset \Delta$, then $\lim_{n\to\infty} f * \phi_n = f$.

Exercise 4.36. — Suppose $\lambda \in \mathscr{H}(A)$. Use the technical lemma above along with Fatou's lemma to show that there is a Haar measure $\mu \in \mathscr{H}(\widehat{A})$ such that for any function $f \in \mathfrak{L}_1(A,\lambda) \cap \mathfrak{L}_2(A,\lambda)$, the Fourier transform $\hat{f}$ is an element of $\mathfrak{C}_0(\widehat{A}) \cap \mathfrak{L}_2(\widehat{A},\mu)$, and

$$\|\hat{f}\|_2 \leq \|f\|_2.$$

Dually, for any function $\phi \in \mathfrak{L}_1(\widehat{A},\mu) \cap \mathfrak{L}_2(\widehat{A},\mu)$, the inverse Fourier transform $\check{\phi}$ is an element of $\mathfrak{C}_0(A) \cap \mathfrak{L}_2(A,\lambda)$, and

$$\|\check{\phi}\|_2 \leq \|\phi\|_2.$$

4.37. — Suppose $\lambda \in \mathscr{H}(A)$, and suppose $\mu \in \mathscr{H}(\widehat{A})$ as above. Then since $\mathfrak{L}_1(A,\lambda) \cap \mathfrak{L}_2(A,\lambda)$ is dense in $\mathfrak{L}_2(A,\lambda)$, the Fourier transform $f \longmapsto \hat{f}$ can be extended uniquely to a linear continuous map $\mathfrak{L}_2(A,\lambda) \longrightarrow \mathfrak{L}_2(\widehat{A},\mu)$. Dually, the inverse Fourier transform $f \longmapsto \check{f}$ can be extended uniquely to a linear continuous map $\mathfrak{L}_2(\widehat{A},\mu) \longrightarrow \mathfrak{L}_2(A,\lambda)$.

Exercise 4.38 **(Fourier inversion formula).** — Use our technical lemma again to show that for any $\lambda \in \mathscr{H}(A)$, there is a unique $\mu \in \mathscr{H}(\widehat{A})$ such that any $f \in \mathfrak{L}_2(A,\lambda)$,

$$\check{\hat{f}} = f,$$

and for any $\phi \in \mathfrak{L}_2(\widehat{A},\mu)$,

$$\hat{\check{\phi}} = \phi.$$

4.39. — Given a Haar measure $\lambda \in \mathscr{H}(A)$, the unique measure $\mu \in \mathscr{H}(\widehat{A})$ for which the Fourier inversion formula obtains is called the *dual measure*. It follows from Pontryagin duality that the formation of the dual measure is a bijection $\mathscr{H}(A) \longrightarrow \mathscr{H}(\widehat{A})$. We will simply refer to elements of the graph of this bijection as *dual pairs of Haar measures on A and $\widehat{A}$.*

Exercise 4.40 **(Plancherel's theorem).** — Suppose (λ,μ) a dual pair of measures on A and $\widehat{A}$. Use the Fourier inversion formula to show that the Fourier transformation is a linear isometry $\mathfrak{L}_2(A,\lambda)\longrightarrow\mathfrak{L}_2(\widehat{A})$ whose inverse is the inverse Fourier transformation.

Exercise 4.41 **(Parseval's identity).** — Show that for any $f_1, f_2 \in \mathfrak{L}_2(A)$, the identity

$$\int_A f_1\overline{f_2}\,d\lambda = \int_{\widehat{A}} \hat{f}_1\overline{\hat{f}_2}\,d\mu.$$

obtains.

Exercise 4.42. — Use Parseval's identity to solve the first riddle posed in 1.1 in the following way. Consider the characteristic function

$$\chi_{[-1,1]}(x) = \begin{cases} 1 & \text{if } |x| \leq 1; \\ 0 & \text{else} \end{cases}$$

on $\mathbf{R}$. Contemplate its Fourier transform and use Parseval's identity to show that

$$\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2} dt = \pi.$$

Exercise 4.43. — Use Parseval's identity to address the other riddle from 1.1. [Hint: the answer is $\pi/4$.]

Exercise 4.44. — For any positive integer m, compute

$$\int_{-\infty}^{\infty} \frac{\sin^{2m} t}{t^{2m}}\, dt.$$

Exercise 4.45. — Now let's discuss the second riddle, 1.2. For any positive integer m, write

$$\zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \frac{1}{2}\sum_{n\in\mathbf{Z}} \frac{1}{n^{2m}}.$$

This suggests we should use Parseval's identity with $A = \mathbf{U}(1)$ and $\widehat{A} = \mathbf{Z}$. Using this thought, show that

$$\zeta(2m) = \frac{(-1)^{m+1}(2\pi)^m B_{2m}}{2(2m)!},$$

where B_{2m} is the $2m$-th *Bernoulli number*. The k-th Bernoulli number is defined as the k-th coefficient in the power series expansion of $t/(e^t-1)$:

$$\frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

They can be described with a simple closed formula:

$$B_k = \sum_{i=0}^{k}\sum_{j=0}^{i} \binom{i}{j} \frac{j^k}{i+1}.$$

We haven't addressed Ramanujan's mysterious remarks. I hope it has piqued your curiosity!

CLARK BARWICK