A CHARACTERIZATION OF SIMPLICIAL LOCALIZATION FUNCTORS AND A DISCUSSION OF DK EQUIVALENCES

C. BARWICK AND D. M. KAN

ABSTRACT. In a previous paper we lifted Charles Rezk's complete Segal model structure on the category of simplicial spaces to a Quillen equivalent one on the category of "relative categories." Here, we characterize simplicial localization functors among relative functors from relative categories to simplicial categories as any choice of homotopy inverse to the delocalization functor of Dwyer and the second author. We employ this characterization to obtain a more explicit description of the weak equivalences in the model category of relative categories mentioned above by showing that *these weak equivalences are exactly the DK-equivalences*, i.e. those maps between relative categories which induce a weak equivalence between their simplicial localizations.

1. An overview

We start with some preliminaries.

1.1. Relative categories. As in [BK] we denote by **RelCat** the category of (small) relative categories and relative functors between them, where by a **relative category** we mean a pair $(\boldsymbol{C}, \boldsymbol{W})$ consisting of a category $\boldsymbol{C}$ and a subcategory $\boldsymbol{W} \subset \boldsymbol{C}$ which contains all the objects of $\boldsymbol{C}$ and their identity maps and of which the maps will be referred to as **weak equivalences** and where by a **relative functor** between two such relative categories we mean a weak equivalence preserving functor.

1.2. Rezk equivalences. In [BK] we lifted Charles Rezk's complete Segal model structure on the category s$\boldsymbol{S}$ of (small) *simplicial spaces* (i.e. bisimplicial sets) to a Quillen equivalent model structure on the category **RelCat** (1.1). We will refer to the weak equivalences in both these model structures as **Rezk equivalences** and denote by both

$$\mathbf{Rk} \subset \mathrm{s}\boldsymbol{S} \qquad \text{and} \qquad \mathbf{Rk} \subset \mathbf{RelCat}$$

the subcategories consisting of these Rezk equivalences.

1.3. Homotopy equivalences between relative categories. A relative functor $f \colon \boldsymbol{X} \to \boldsymbol{Y}$ between two relative categories (1.1) is called a **homotopy equivalence** if there exists a relative functor $g \colon \boldsymbol{Y} \to \boldsymbol{X}$ (called a **homotopy inverse** of f) such that the compositions gf and fg are naturally weakly equivalent (i.e. can be connected by a finite zigzag of natural weak equivalences) to the identity functors of $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively.

1.4. **DK-equivalences.** A map in the category **SCat** of *simplicial categories* (i.e. categories enriched over simplicial sets) is [Be1] called a DK-**equivalence** if it induces *weak equivalences* between the simplicial sets involved and an *equivalence of categories* between their *homotopy categories*, i.e. the categories obtained from them by replacing each simplicial set by the set of its components.

Furthermore a map in **RelCat** will similarly be called a DK-**equivalence** if its image in **SCat** is so under the *hammock localization functor* [DK2]

$$L^H \colon \textbf{RelCat} \longrightarrow \textbf{SCat}$$

(or of course the naturally DK-equivalent functors **RelCat** → **SCat** considered in [DK1] and [DHKS, 35.6]).

We will denote by both

$$\textbf{DK} \subset \textbf{SCat} \qquad \text{and} \qquad \textbf{DK} \subset \textbf{RelCat}$$

the subcategories consisting of these DK-equivalences.

Next we define what we mean by

1.5. **Simplicial localization functors.** In defining DK-equivalences in **RelCat** (1.4) we used the hammock localization functor and not one of the other DK-equivalent functors mentioned because, for our purposes here it seemed to be the more convenient one. However in other situations the others are more convenient and it therefore makes sense to define in general a **simplicial localization functor** as any functor **RelCat** → **SCat** which is naturally DK-equivalent to the functors mentioned above (1.4).

We also need

1.6. **The relativization functor.** In contrast with the situation mentioned in 1.5 there *is* a preferred choice for a **relativization functor**

$$R \colon \textbf{SCat} \longrightarrow \textbf{RelCat}$$

which is a kind of inverse of the simplicial localization functor, namely the *delocalization* mentioned in [DK3, 2.5] which assigns to an object $\textbf{\textit{A}} \in \textbf{SCat}$ its relative *flattening* which is the relative category which consists of

   (i) a category which is the Grothendieck construction on $\textbf{\textit{A}}$, where $\textbf{\textit{A}}$ is considered as a simplicial diagram of categories, and
   (ii) its subcategory obtained by applying the same construction to the subobject of $\textbf{\textit{A}}$ which consists of its objects only.

Our first main result then is

1.7. **Theorem.** *A relative functor*

$$(\textbf{RelCat}, \textbf{DK}) \longrightarrow (\textbf{SCat}, \textbf{DK})$$

is a simplicial localization functor (1.5) *iff it is a homotopy inverse* (1.3) *of the realization functor* (1.6)

$$\mathrm{Rel} \colon (\textbf{SCat}, \textbf{DK}) \longrightarrow (\textbf{RelCat}, \textbf{DK}) \ .$$

Our second main result then is

1.8. Theorem. *A map in* **RelCat** *(1.1) is a Rezk equivalence (1.2) iff it is a DK-equivalence (1.4).*

We offer some

1.9. Comments on the proof of 1.7. The proof of theorem 1.7 heavily involves some of the results of [DK1] and [DK3, 2.5] and we therefore first (in §2) review some of the results of these papers.

In §3 we then actually prove theorem 1.7. It turns out however that in addition to the results mentioned in §2 we need a property of the hammock localization of which we will give two proofs. The first is a very short one based on a remark of Toen and Vezzosi [TV, 2.2.1] involving the *homotopy category* of **SCat**. The other, which is due to Bill Dwyer, relies heavily on [DK1] and [DK2] and is longer, but has the "advantage" of taking place in the *model category* itself.

We end with some

1.10. Comments on the proof of 1.8. The proof of theorem 1.8 uses three key facts:

   (i) If $f\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a homotopy equivalence between relative categories that have the two out of three property, then a map $x\colon X_1 \to X_2 \in \boldsymbol{X}$ is a weak equivalence (1.1) iff the induced map $fx\colon fX_1 \to fX_2 \in \boldsymbol{Y}$ is so.
   (ii) In view of [BK, 6.1], the simplicial nerve functor $N\colon \mathbf{RelCat} \to \mathrm{s}\boldsymbol{S}$ (4.1) is a homotopy equivalence (1.2)

$$N\colon (\mathbf{RelCat}, \mathbf{Rk}) \longrightarrow (\mathrm{s}\boldsymbol{S}, \mathbf{Rk})\ .$$

   (iii) In view of 1.7, the relativization functor Rel$\colon$ **SCat** $\to$ **RelCat** (1.6) is a homotopy equivalence

$$\mathrm{Rel}\colon (\mathbf{SCat}, \mathbf{DK}) \longrightarrow (\mathbf{RelCat}, \mathbf{DK})\ .$$

   (iv) In view of [Be2, 6.3 and 8.6], the flipped nerve functor $Z\colon$ **SCat** $\to$ s$\boldsymbol{S}$ (4.2) is a homotopy equivalence

$$Z\colon (\mathbf{SCat}, \mathbf{DK}) \longrightarrow (\mathrm{s}\boldsymbol{S}, \mathbf{Rk})\ .$$

These results strongly suggest that theorem 1.8 should be true. To complete the proof, one just has to show that the functors N Rel and $Z\colon$ **SCat** $\to$ s$\boldsymbol{S}$ are naturally Rezk equivalent. In fact we will prove the following somewhat stronger result:

1.11. Proposition. *The functors*

$$N\,\mathrm{Rel}\ \ and\ Z\colon \mathbf{SCat} \longrightarrow \mathrm{s}\boldsymbol{S}$$

are naturally Reedy equivalent.

This will be established in §4.

2. Preliminaries for theorem 1.7

In preparation for the proof (in §3) of theorem 1.7 we review here some of the results of [DK1], [DK2] and [DK3, 2.3] which will be needed.

2.1. **The hammock localization.** In the proof of 1.7 we will make extensive use of the hammock localization L^H of [DK2] because, unlike the other simplicial localization functors, it has the property that every relative category $(\boldsymbol{C}, \boldsymbol{W})$ comes with a *natural embedding* $\boldsymbol{C} \to L^H(\boldsymbol{C}, \boldsymbol{W})$.

2.2. **The category RelSCat.** This will be the category which has as its objects the pairs $(\boldsymbol{A}, \boldsymbol{U})$ where $\boldsymbol{A} \in \mathbf{SCat}$ and $\boldsymbol{U} \subset \boldsymbol{A}$ is a subobject which contains all the objects of $\boldsymbol{A}$.

One then can consider **RelCat** as a full subcategory of **RelSCat** and [DK2, 2.5] extend the functor $L^H \colon \mathbf{RelCat} \to \mathbf{SCat}$ to a functor $L^H \colon \mathbf{RelSCat} \to \mathbf{SCat}$ by sending an object of **RelSCat** to the diagonal of the bisimplicial set obtained from it by dimensionwise application of the hammock localization.

To deal with [DK1] and [DK3, 2.5] it will be convenient to introduce a notion of

2.3. **Neglectable categories.** Given an object $(\boldsymbol{A}, \boldsymbol{U}) \in \mathbf{RelSCat}$ (2.2) we will say that $\boldsymbol{U}$ is *neglectable* in $\boldsymbol{A}$ if every map of $\boldsymbol{U}$ goes to an *isomorphism* in $\pi_0 \boldsymbol{A}$.

2.4. **Some results from [DK1].** [DK1, 3.4 and 5.1] then imply

(i) *Let $\boldsymbol{A}$ be a category, let $\boldsymbol{U}$ and $\boldsymbol{V} \subset \boldsymbol{A}$ be subcategories which contain all the objects of $\boldsymbol{A}$ and let $\boldsymbol{U} \cup \boldsymbol{V} \subset \boldsymbol{A}$ denote the subcategory spanned by $\boldsymbol{U}$ and $\boldsymbol{V}$ and assume that $\boldsymbol{V}$ is neglectable in $L^H(\boldsymbol{A}, \boldsymbol{U})$. Then the induced map*

$$L^H(\boldsymbol{A}, \boldsymbol{U}) \longrightarrow L^H(\boldsymbol{A}, \boldsymbol{U} \cup \boldsymbol{V}) \in \mathbf{SCat}$$

is a DK-equivalence.

Similarly [DK1, 6.4] implies

(ii) *Let $(\boldsymbol{B}, \boldsymbol{V}) \in \mathbf{RelSCat}$ be such that $\boldsymbol{V}$ is neglectable in $\boldsymbol{B}$. Then the induced map (2.1 and 2.2)*

$$\boldsymbol{B} \longrightarrow L^H(\boldsymbol{B}, \boldsymbol{V}) \in \mathbf{SCat}$$

is a DK-equivalence.

We end with a brief review of

2.5. **The relativization functor [DK3, 2.5].** The **relativization functor** is the functor

$$\mathrm{Rel} \colon \mathbf{SCat} \longrightarrow \mathbf{RelCat}$$

which sends an object $\boldsymbol{A} \in \mathbf{SCat}$ to the object $(b\boldsymbol{A}, b\mathbf{id}) \in \mathbf{RelCat}$, where $b\boldsymbol{A}$ is the *flattening* of $\boldsymbol{A}$, i.e. the category which has as objects the pairs (A, n), where A is an object of $\boldsymbol{A}$ and n is an integer ≥ 0 and which has as maps $(A_1, n_1) \to (A_2, n_2)$ the pairs (a, q) where a is a map $A_1 \to A_2 \in \boldsymbol{A}_{n_2}$ and q is a simplicial operator from dimension n_1 to dimension n_2 and $b\mathbf{id} \in b\boldsymbol{A}$ is the subcategory consisting of the maps (a, q) for which a is an identity map.

It then was noted in [DK3, 2.5] that, for every object $\boldsymbol{A} \in \mathbf{SCat}$, there exists an object $\overline{\boldsymbol{A}} \in \mathbf{SCat}$ with the same object set as $b\boldsymbol{A}$ with the following properties:

(i) *There is a natural monomorphism $\boldsymbol{A} \to \overline{\boldsymbol{A}}$ which is a DK-equivalence.*

(ii) *There is a natural embedding $b\boldsymbol{A} \to \overline{\boldsymbol{A}}$ with the property that (if the image of $b\mathbf{id} \in b\boldsymbol{A}$ in $\overline{\boldsymbol{A}}$ is also denoted by $b\mathbf{id}$) the induced map*

$$\mathrm{L}^{\mathrm{H}}(b\boldsymbol{A}, b\mathbf{id}) \longrightarrow \mathrm{L}^{\mathrm{H}}(\overline{\boldsymbol{A}}, b\mathbf{id}) \in \mathbf{SCat}$$

is a DK-equivalence.

(iii) $b\mathbf{id}$ *is neglectable in $\overline{\boldsymbol{A}}$ (2.3) and hence the embedding (2.1 and 2.2)*

$$\overline{\boldsymbol{A}} \longrightarrow \mathrm{L}^{\mathrm{H}}(\overline{\boldsymbol{A}}, b\mathbf{id}) \in \mathbf{SCat}$$

is a DK-equivalence.

It follows that

(iv) $\boldsymbol{A}$ *and $\mathrm{L}^{\mathrm{H}} \operatorname{Rel} \boldsymbol{A}$ can be connected by the natural zigzag of DK-equivalences*

$$\boldsymbol{A} \longrightarrow \overline{\boldsymbol{A}} \longrightarrow \mathrm{L}^{\mathrm{H}}(\overline{\boldsymbol{A}}, b\mathbf{id}) \longleftarrow \mathrm{L}^{\mathrm{H}}(b\boldsymbol{A}, b\mathbf{id}) = \mathrm{L}^{\mathrm{H}} \operatorname{Rel} \boldsymbol{A}$$

which in turn implies that

(v) Rel *is a relative functor* (1.4)

$$\operatorname{Rel} \colon (\mathbf{SCat}, \mathbf{DK}) \longrightarrow (\mathbf{RelCat}, \mathbf{DK}) \ .$$

3. A PROOF OF THEOREM 1.7

To prove theorem 1.7 is suffices, in view of 2.5(iv) and (v), to prove

3.1. **Proposition.** *Every object $(\boldsymbol{C}, \boldsymbol{W}) \in \mathbf{RelCat}$ is naturally DK-equivalent to* $\operatorname{Rel} \mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W})$.

Proof. Consider the commutative diagram in **RelSCat** (2.2)

in which

- c is as in 2.1, f is as in 2.5(i), d and e are as in 2.5(ii) and a is the unique map such that $da = fc$, and
- the symbol $\cup$ is as in 2.4(i) and, in the formulas which involve two $\boldsymbol{W}$'s, the second $\boldsymbol{W}$ is the image of the $\boldsymbol{W}$ in the upper left $(\boldsymbol{C}, \boldsymbol{W})$.

Then it suffices to show that a and b are DK-equivalences in **RelCat** or equivalently that $\mathrm{L}^{\mathrm{H}} a$ and $\mathrm{L}^{\mathrm{H}} b$ are DK-equivalences in **SCat**.

This is done as follows:

The map f admits a factorization

$$\left(\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}), \boldsymbol{W}\right) \xrightarrow{\ x\ } \left(\overline{\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W})}, \boldsymbol{W}\right) \xrightarrow{\ y\ } \left(\overline{\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W})}, b\mathbf{id} \cup \boldsymbol{W}\right)$$

in which clearly $\boldsymbol{W}$ is neglectable (2.3) in $\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W})$ and hence, in view of 2.4(ii) and 2.5(i), $\mathrm{L}^{\mathrm{H}} x$ is a DK-equivalence and $\boldsymbol{W}$ is neglectable in $\overline{\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W})}$. It follows that (2.5(iii)) $\mathrm{L}^{\mathrm{H}} y$ is a DK-equivalence and hence (2.4(ii)) so is $\mathrm{L}^{\mathrm{H}} g$.

Furthermore, in view of 2.5(ii), $\mathrm{L}^{\mathrm{H}} e$ is a DK-equivalence and consequently $\mathrm{L}^{\mathrm{H}} d$ is a DK-equivalence and $\boldsymbol{W}$ is neglectable in $\mathrm{L}^{\mathrm{H}}\bigl(b\,\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}), b\mathbf{id}\bigr)$ which implies that $\mathrm{L}^{\mathrm{H}} b$ is a DK-equivalence.

It thus remains to prove that $\mathrm{L}^{\mathrm{H}} a$ is a DK-equivalence, but this now follows from

3.2. Proposition. $\mathrm{L}^{\mathrm{H}} c\colon \mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}) \to \mathrm{L}^{\mathrm{H}}\bigl(\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}), \boldsymbol{W}\bigr)$ *is a DK-equivalence.*

We will give two proofs of this proposition. The first is short and is based on a remark by Toen and Vezzosi involving the *homotopy category* $\mathrm{Ho}\,\mathbf{SCat}$ of $\mathbf{SCat}$. The other, due to Bill Dwyer, is longer but takes place inside the *model category* $\mathbf{S}O\text{-}\mathbf{Cat}$ of the simplicial categories with a fixed object set O (in this case the object set of $\boldsymbol{C}$).

They both involve the commutative diagram

(i)
$$
\begin{array}{ccc}
\boldsymbol{C} & \longrightarrow & \mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}) \\
\downarrow & & \downarrow \\
\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}) & \xrightarrow{\ \mathrm{L}\,c\ } & \mathrm{L}^{\mathrm{H}}\bigl(\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}), \boldsymbol{W}\bigr)
\end{array}
$$
in which the unmarked maps are as in 2.1, and
(ii) in which, in view of 2.4(ii) the right hand map is a DK-equivalence.

3.3. The short proof. In view of [TV, 2.2.1] the maps at the right and the bottom in 3.2(i) have the same image in $\mathrm{Ho}\,\mathbf{SCat}$ and as (3.2(ii)) the one on the right is a DK-equivalence, so is the one at the bottom.

3.4. The longer proof. We start with a brief discussion of

(i) **Homotopy pushouts**

Given a model category together with a choice of *cofibrant replacement functor* and a choice of *functorial factorization* of maps into a cofibration followed by a trivial fibration, associate with every zigzag $Y \leftarrow X \to Z$ a commutative diagram

$$
\begin{array}{ccccc}
Y^{\mathrm{c}} & \longleftarrow & X^{\mathrm{c}} & \longrightarrow & Z^{\mathrm{c}} \\
\sim\downarrow & \searrow & \downarrow & \swarrow & \downarrow\sim \\
Y^{\mathrm{c}'} & & & & Z^{\mathrm{c}'} \\
\sim\downarrow & & \sim\downarrow & & \downarrow\sim \\
Y & \longleftarrow & X & \longrightarrow & Z
\end{array}
$$

as follows. The pentagon is obtained by applying the cofibrant approximation functor and the two triangles by means of the functorial factorization. Consequently the maps indicated $\sim$ are weak equivalences.

Then the pushout $Y^{\mathrm{c}} \amalg_{X^{\mathrm{c}}} Z^{\mathrm{c}}$ is a homotopy pushout of the zigzag $Y \leftarrow X \to Z$.

Clearly this construction is functorial in the sense that every diagram of the form

$$
\begin{array}{ccccc}
Y_0 & \longleftarrow & X_0 & \longrightarrow & Z_0 \\
\downarrow{\scriptstyle y} & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle z} \\
Y_1 & \longleftarrow & X_1 & \longrightarrow & Z_1
\end{array}
$$

induces a map

$$
Y_0^{\mathrm{c}} \amalg_{X_0^{\mathrm{c}}} Z_0^{\mathrm{c}} \longrightarrow Y_1^{\mathrm{c}} \amalg_{X_1^{\mathrm{c}}} Z_1^{\mathrm{c}}
$$

which is a weak equivalence whenever y, x and z are.

Next we discuss

(ii) **The original simplicial localization functor** L [DK1]

We will work in the model category $\mathbf{S}O\text{-}\mathbf{Cat}$ [DK1] of the simplicial categories with a fixed object set O (which will be the object set of $\boldsymbol{C}$). The weak equivalences in the model structure are the DK-equivalences.

As [DK1, 4.1] $\mathrm{L}(\boldsymbol{C}, \boldsymbol{W})$ is the pushout of the zigzag

$$
\mathrm{F}_* \boldsymbol{C} \longleftarrow \mathrm{F}_* \boldsymbol{W} \longrightarrow \mathrm{F}_* \boldsymbol{W}[\mathrm{F}_* \boldsymbol{W}^{-1}]
$$

and the map $\mathrm{F}_* \boldsymbol{W} \to \mathrm{F}_* \boldsymbol{C}$ is a cofibration, it follows from [DK1, 8.1] that this pushout is also a homotopy pushout. Consequently

(∗) $\mathrm{L}(\boldsymbol{C}, \boldsymbol{W})$ is naturally DK-equivalent to

$$
(\mathrm{F}_* \boldsymbol{C})^{\mathrm{c}} \amalg_{(\mathrm{F}_* \boldsymbol{W})^{\mathrm{c}}} \left(\mathrm{F}_* \boldsymbol{W}[\mathrm{F}_* \boldsymbol{W}^{-1}] \right)^{\mathrm{c}} \ .
$$

Now we turn to

(iii) **The hammock localization** L^{H} [DK2]

In view of [DK2, 2.5] the functors L and L^{H} are naturally DK-equivalent and there exists a diagram of the form

$$
\begin{array}{ccccccc}
\mathrm{F}_* \boldsymbol{C} & \longleftarrow & \mathrm{F}_* \boldsymbol{W} & \longrightarrow & \mathrm{F}_* \boldsymbol{W}[\mathrm{F}_* \boldsymbol{W}^{-1}] & = & \mathrm{L}(\boldsymbol{W}, \boldsymbol{W}) \\
{\scriptstyle \approx}\uparrow & & {\scriptstyle \approx}\uparrow & & \uparrow & & \\
\mathrm{F}_* \boldsymbol{C} & \longleftarrow & \mathrm{F}_* \boldsymbol{W} & \longrightarrow & \mathrm{L}^{\mathrm{H}}(\mathrm{F}_* \boldsymbol{W}, \mathrm{F}_* \boldsymbol{W}) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\boldsymbol{C} & \longleftarrow & \boldsymbol{W} & \longrightarrow & \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W}) & &
\end{array}
$$

in which the vertical maps are DK-equivalences. Hence

(∗) $\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W})$ is naturally DK-equivalent to

$$
\boldsymbol{C}^{\mathrm{c}} \amalg_{\boldsymbol{W}^{\mathrm{c}}} \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}}
$$

and $\mathrm{L}^{\mathrm{H}}\big(\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}), \boldsymbol{W}\big)$ is naturally DK-equivalent to

$$
\boldsymbol{C}^{\mathrm{c}} \amalg_{\boldsymbol{W}^{\mathrm{c}}} \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}} \amalg_{\boldsymbol{W}^{\mathrm{c}}} \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}}
$$

in which the two middle maps $\boldsymbol{W}^{\mathrm{c}} \to \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}}$ are the same and which therefore is the same as

$$\boldsymbol{Q} = \mathrm{colim} \left(\boldsymbol{C}^{\mathrm{c}} \longleftarrow \boldsymbol{W}^{\mathrm{c}} \begin{array}{c} \nearrow \; \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}} \\ \searrow \; \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}} \end{array} \right) \; .$$

It follows that diagram 3.2(i) is DK-equivalent to the commutative diagram

$$\begin{array}{ccc} \boldsymbol{C}^{\mathrm{c}} & \longrightarrow & \boldsymbol{C}^{\mathrm{c}} \amalg_{\boldsymbol{W}^{\mathrm{c}}} \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}} \\ \downarrow & & \downarrow u \\ \boldsymbol{C}^{\mathrm{c}} \amalg_{\boldsymbol{W}^{\mathrm{c}}} \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}} & \xrightarrow{\;\;v\;\;} & \boldsymbol{Q} \end{array}$$

in which u is obtained by mapping the zigzag $\boldsymbol{C}^{\mathrm{c}} \leftarrow \boldsymbol{W}^{\mathrm{c}} \to \mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}}$ to the upper zigzag in the diagram whose colimit is $\boldsymbol{Q}$ and v is obtained by mapping it to the lower zigzag.

In view of 3.2(ii) the map u is a DK-equivalence and as $v = Tu$ where $T \colon \boldsymbol{Q} \to \boldsymbol{Q}$ denotes the automorphism which switches the two copies of $\mathrm{L}^{\mathrm{H}}(\boldsymbol{W}, \boldsymbol{W})^{\mathrm{c}}$, so is the map v and therefore also the desired map

$$\mathrm{L}\, c \colon \; \mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}) \longrightarrow \mathrm{L}^{\mathrm{H}}\big(\mathrm{L}^{\mathrm{H}}(\boldsymbol{C}, \boldsymbol{W}), \boldsymbol{W}\big) \; .$$

4. Completion of the proof of 1.11

Before completing the proof of theorem 1.8, i.e. proving proposition 1.11, we recall first some of the notions involved.

4.1. The simplicial nerve functor N. This is the functor $N \colon \mathbf{RelCat} \to \mathrm{s}\boldsymbol{S}$ which sends an object $X \in \mathbf{RelCat}$ to the bisimplicial set which has as its (p, q)-bisimplices $(p, q \geq 0)$ the maps

$$\check{p} \times \hat{q} \longrightarrow X \in \mathbf{RelCat}$$

where $\check{p}$ denotes the category $0 \to \cdots \to p$ in which only the identity maps are weak equivalences and $\hat{q}$ denotes the category $0 \to \cdots \to q$ in which all maps are weak equivalences.

4.2. The flipped nerve functor Z. This is the functor $Z \colon \mathbf{SCat} \to \mathrm{s}\boldsymbol{S}$ which sends an object $\boldsymbol{A} \in \mathbf{SCat}$ to the simplicial space $Z\boldsymbol{A}$ of which the space in dimension $k \geq 0$ is the simplicial set $(ZA)_k$ which is the disjoint union, taken over all ordered sequences $A_0, \ldots, A_k$ of objects of $\boldsymbol{A}$, of the products

$$\mathrm{hom}(A_0, A_1) \times \cdots \times \mathrm{hom}(A_{k-1}, A_k) \; .$$

4.3. The opposite Γ^{op} of the category of simplices functor Γ. This is the functor $\Gamma^{\mathrm{op}} \colon \boldsymbol{S} \to \mathbf{Cat}$ which sends a simplicial set $X \in \boldsymbol{S}$ to its category of simplices, i.e. the category which has

(i) as objects the pair (p, x) consisting of an integer $p \geq 0$ and a p-simplex of X, and
(ii) as maps $(p_1, x_1) \to (p_2, x_2)$ the simplicial operators t from dimension p_1 to dimension p_2 such that $tx_1 = x_2$.

We also need

4.4. **Some auxiliary notions.** For every object $A \in \mathbf{SCat}$, denote

- by YA the simplicial diagram of categories of which the category $(YA)_k$ in dimension $k \geq 0$ has as objects the sequences of maps in bA (1.6) of the form

$$(p_0, A_0) \xrightarrow{(t_1, a_1)} \cdots \xrightarrow{(t_k, a_k)} (p_k, A_k)$$

and as maps the commutative diagrams in bA of the form

$$
\begin{array}{ccc}
(p_0, A_0) \xrightarrow{(t_1, a_1)} \cdots \xrightarrow{(t_k, a_k)} (p_k, A_k) \\
{\scriptstyle (u_0, \mathbf{id})}\downarrow \qquad\qquad\qquad\qquad \downarrow{\scriptstyle (u_k, \mathbf{id})} \\
(p_0', A_0) \xrightarrow{(t_1', a_1')} \cdots \xrightarrow{(t_k', a_k')} (p_k', A_k)
\end{array}
$$

and

- by $\overline{YA} \subset YA$ the subobject of which the category $(\overline{YA})_k$ in dimension k is the subcategory of $(YA)_k$ consisting of the above maps for which *the t_i's and the t_i''s are identities and hence all p_i's are the same, all p_i''s are the same and all u_i's are the same.*

Then there is a *strong deformation retraction* of YA onto $\overline{YA}$ which to each object of YA as above assigns the map

$$
\begin{array}{ccc}
(p_0, A_0) \xrightarrow{\quad (t_1, a_1) \quad} \cdots \xrightarrow{\quad (t_k, a_k) \quad} (p_k, A_k) \\
{\scriptstyle (t_k \cdots t_1, \mathbf{id})}\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle (\mathbf{id}, \mathbf{id})} \\
(p_k, A_0) \xrightarrow{(\mathbf{id}, t_k \cdots t_1 a_1)} \cdots \xrightarrow{\quad (\mathbf{id}, a_k) \quad} (p_k, A_k)
\end{array}
$$

the existence of which implies that

(i) *the inclusion $\overline{YA} \subset YA$ is a dimensionwise weak equivalence of categories.*

One also readily verifies that there is a canonical 1-1 correspondence between the objects of $(\overline{YA})_k$ and the simplices of $(ZA)_k$ (4.2) and that in effect

(ii) *there is a canonical isomorphism $\overline{YA} \approx \Gamma^{\mathrm{op}} ZA$ (4.3).*

Now we are ready for the

4.5. **Completion of the proof.** Let $n \colon \mathbf{Cat} \to S$ denote the *classical nerve functor*.

Then clearly $N \operatorname{Rel} A = nYA$ (4.4) and if one defines $\overline{N \operatorname{Rel} A} \subset N \operatorname{Rel} A$ by $\overline{N \operatorname{Rel} A} = n\overline{YA}$, then it follows from 4.4(i) that

(i) *the inclusion $\overline{N \operatorname{Rel} A} \to N \operatorname{Rel} A \in \mathrm{s}S$ is a Reedy equivalence.*

Moreover it follows from 4.4(ii) that

(ii) *there is a canonical isomorphism $\overline{N \operatorname{Rel} A} \approx n\Gamma^{\mathrm{op}} ZA$*

and to complete the proof of proposition 1.11 and hence of theorem 1.8 it thus suffices, in view of the fact that clearly

(iii) *the functors $n\Gamma^{\mathrm{op}}$ and $n\Gamma \colon S \to S$ (4.3) are naturally weakly equivalent,*

to show that

(iv) *there exists a natural Reedy equivalence $n\Gamma ZA \to ZA \in \mathrm{s}S$.*

But this follows immediately from the observation of Dana Latch [L] (see also [H, 18.9.3]) that there exists a natural weak equivalence

$$n\Gamma \longrightarrow 1 \ .$$

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139

   E-mail address: clarkbar@gmail.com

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139