APPLICATIONS OF DERIVED ALGEBRAIC GEOMETRY TO HOMOTOPY THEORY

by

Clark Barwick

The method of "postulating" what we want has many advantages;
they are the same as the advantages of theft over honest toil. — B. RUSSELL

These are notes for a minicourse at the "i-MATH School of Derived Algebraic Geometry," 1-6 June 2009, at the Departamento de Matemáticas at the Universidad de Salamanca. The standard caveats apply here: (1) These notes are very informal, and even when I'm giving details, I'm skipping details. (2) None of the ideas are mine, and all interesting results should be ascribed to others. (3) All errors are mine, and I'm duly ashamed.

Contents

1. The geometry of E_∞ ring spectra

Spectra. — Topological algebraic geometry begins with a rewriting of the notion of abelian group. The analogue of the notion of an abelian group in homotopy theory is the notion of a *spectrum*.

1.1. — Recall that a *cohomology theory* E^* is a collection of contravariant functors E^n — one for every integer n — from the category of pairs of spaces (X,Y) (with $Y \subset X$) to the category of abelian groups, along with natural transformations

$$\delta^n : E^n(Y,Z) \longrightarrow E^{n+1}(X,Y),$$

functorial in triples $Z \subset Y \subset X$, subject to the following conditions. (Here one writes $E^n(X)$ for $E^n(X,\varnothing)$.)

(1.1.1) For any weak equivalence of spaces $f : X' \longrightarrow X$, the induced homomorphism

$$f^\star : E^n(X) \longrightarrow E^n(X')$$

is an isomorphism.

(1.1.2) If $X = \coprod_\alpha X_\alpha$, then the induced homomorphisms

$$E^n(X) \longrightarrow \prod_\alpha E^n(X_\alpha)$$

are isomorphisms.

(1.1.3) For every triple $Z \subset Y \subset X$, one has a long exact sequence

$$\dots \longrightarrow E^n(X,Y) \longrightarrow E^n(X,Z) \longrightarrow E^n(Y,Z) \xrightarrow{\delta^n} E^{n+1}(X,Y) \longrightarrow \dots.$$

(1.1.4) If $Y \subset X$, and if U is an open subspace such that the closure of U is contained in the interior of Y, then the induced homomorphisms

$$E^n(X,Y) \longrightarrow E^n(X-U, Y-U)$$

are isomorphisms.

Example 1.2. — (1.2.1) Suppose A is an abelian group. Then there is *singular cohomology* HA^* with

$$(HA)^n(X,Y) := H^n(X,Y;A).$$

This is the unique cohomology theory that satisfies the so-called *dimension axiom*:

$$(HA)^n(\star) = \begin{cases} A & \text{if } n = 0; \\ 0 & \text{else.} \end{cases}$$

(1.2.2) Then there is *complex K-theory* KU^*. For finite cell complexes X, the abelian group $KU^0(X)$ is the Grothendieck group of stable isomorphism classes of complex vector bundles on X. The dimension axiom definitely is not satisfied for KU:

$$KU^n(\star) = \begin{cases} \mathbf{Z} & \text{if } n = 0 \mod 2; \\ 0 & \text{else.} \end{cases}$$

This is the celebrated *Bott periodicity theorem*.

Definition 1.3. — The ∞-category $\mathcal{S}p$ of *spectra* is the homotopy limit of the sequence of ∞-categories

$$\dots \xrightarrow{\Omega} \mathcal{S}_\star \xrightarrow{\Omega} \mathcal{S}_\star \xrightarrow{\Omega} \mathcal{S}_\star \xrightarrow{\Omega} \dots.$$

Expressed differently, a *spectrum* E is a sequence of spaces $\{E(j)\}_{j\in\mathbf{Z}}$ along with weak equivalences

$$E(j) \longrightarrow \Omega E(j+1).$$

1.4. — One may form *homotopy groups* of spectra: for any integer n,

$$\pi_n E := \operatorname{colim}_k \pi_{n+k} E(k).$$

It turns out that a morphism $E \longrightarrow F$ is a weak equivalence if and only if it induces an isomorphism on all homotopy groups.

Evaluation at 0 gives functor $\Omega^\infty : \mathcal{S}p \longrightarrow \mathcal{S}_\star$; it has a left adjoint $\Sigma^\infty : \mathcal{S}_\star \longrightarrow \mathcal{S}p$. For any pointed space X, this gives a spectrum $\Sigma^\infty X$ with the property that

$$\pi_n \Sigma^\infty X = \operatorname{colim}_k \pi_{n+k} \Sigma^n X.$$

The homotopy category $\mathrm{Ho}\,\mathcal{S}p$ of spectra is a triangulated category. In particular, one can shift spectra: if E is a spectrum, then for any integer n, one may form a new spectrum by the formula

$$E[n](j) := E(j-n).$$

There is a t-structure on the homotopy category $\mathrm{Ho}\,\mathcal{S}p$: let $\mathrm{Ho}_{\leq m}\,\mathcal{S}p$ be the full subcategory of spectra E such that each $E(j)$ is $(j+m)$-truncated, and let $\mathrm{Ho}_{\geq m}\,\mathcal{S}p$ be the full subcategory of spectra E such that each $E(j)$ is $(j+m)$-connective. This t-structure is complete, and the heart $\mathcal{S}p^{\heartsuit}$ is canonically equivalent to the category of abelian groups.

***Theorem 1.5* (Brown Representability).** — *Spectra and cohomology theories are the same thing. More precisely, for any spectrum E, there is an associated cohomology theory E^*, defined by the rule*

$$E^n(X) := \mathrm{Mor}_{\mathrm{Ho}\,\mathcal{S}p}(\Sigma^\infty X_+, E[n]). \tag{1.5.1}$$

Moreover, this is the only manner in which cohomology theories arise: for any cohomology theory E^*, there is a (necessarily essentially unique) spectrum E such that (1.5.1) *holds.*

Example 1.6. — (1.6.1) For any abelian group A and any integer n, one can form a space $K(A,n)$ with the property that

$$\pi_j K(A,n) = \begin{cases} A & \text{if } j = n; \\ 0 & \text{else.} \end{cases}$$

It is easy to see that there are canonical weak equivalences $K(A,n) \simeq \Omega K(A,n+1)$. This gives a spectrum HA, whose corresponding cohomology theory $HA^\star$ is singular cohomology. One has

$$\pi_j(HA) = \begin{cases} A & \text{if } j = 0; \\ 0 & \text{else.} \end{cases}$$

(1.6.2) For complex K-theory, there is a spectrum representing KU^*, which has homotopy

$$\pi_j KU = \begin{cases} \mathbf{Z} & \text{if } j = 0 \mod 2; \\ 0 & \text{else.} \end{cases}$$

(1.6.3) The suspension spectrum $\mathbf{S} := \Sigma^\infty S^0$ is the *sphere spectrum*, which is an eye-poppingly important object. Its homotopy groups — the *stable homotopy groups of spheres* — are only known in low degrees. Here are the first few groups:

$$\mathbf{Z} \quad \mathbf{Z}/2 \quad \mathbf{Z}/2 \quad \mathbf{Z}/24 \quad 0 \quad 0 \quad \mathbf{Z}/2 \quad \mathbf{Z}/240 \quad \mathbf{Z}/2 \oplus \mathbf{Z}/2 \quad \ldots$$

1.7. — The last example is the "free spectrum on one generator," and it emphasizes an important feature of homotopy theory in general and topological algebraic geometry in particular: *free objects are complicated*. In most of algebra, we are used to free objects being relatively simple gadgets (though free Lie algebras are no picnic). In homotopy theory, free objects are often the most complicated entities around.

1.8. — We are primarily interested in *multiplicative cohomology theories E^**, which have the special property that, in addition to being a graded abelian group, $E^*(X)$ comes equipped with the structure of a graded ring. In fact, we will ask that it be *graded commutative*.

But where can such a structure come from at the level of spectra?

E_∞ ring spectra. — If spectra replace abelian groups in our story, what should replace commutative rings? To answer this, we must answer two questions: first, what is the "tensor product" of two spectra, and second, what is a "commutative monoid" relative to this tensor product? These questions have generated quite a lot of mathematics, and the answer we give here will just be an overview.

1.9. — The ∞-category $\mathcal{S}p$ of spectra has a unique monoidal structure — the *smash product* $- \wedge -$ — with the following properties.

(1.9.1) The sphere spectrum $\mathbf{S}$ is the unit object.
(1.9.2) The smash product preserves (homotopy) colimits in each variable.

Here's one way to think about this symmetric monoidal structure: Consider the ∞-category $\mathrm{Fun}^L(\mathscr{S}p, \mathscr{S}p)$ of (homotopy) colimit-preserving functors $\mathscr{S}p \longrightarrow \mathscr{S}p$. Composition clearly defines a monoidal structure on this ∞-category; moreover, evaluation at the sphere spectrum $\mathbf{S}$ defines an equivalence of ∞-categories

$$\mathrm{Fun}^L(\mathscr{S}p, \mathscr{S}p) \longrightarrow \mathscr{S}p,$$

so we can just lift the monoidal structure from $\mathrm{Fun}^L(\mathscr{S}p, \mathscr{S}p)$ to $\mathscr{S}p$, and it satisfies our two conditions.

Unfortunately, it's kind of tricky to see directly that this monoidal structure is actually *symmetric*; in fact, historically, it has never been difficult to construct a monoidal structure on $\mathscr{S}p$, but the symmetry has always proven to be a thorny issue. It's actually a deep fact that $\mathscr{S}p$ is a *symmetric monoidal ∞-category*.

1.10. — Now that we have our topological analogue of the tensor product of abelian groups, we wish to formulate a suitably homotopical notion of commutative ring. Here are two strategies that do not work:

(1.10.1) One could opt to take an ordinary symmetric monoidal category that models that symmetric monoidal ∞-category of spectra, and study commutative monoid objects there. Unless you're very clever about how you choose your model, this isn't going to give you the right answer. (This sort of cleverness has been achieved in the models of Elmendorf–Kriz–Mandell–May, Lydakis–Schwede, and Hovey–Schwede–Shipley–Smith.) The point is that it is unreasonable in homotopy theory to expect strictly commutative, unital, and associative ring objects; this is too strong, and it eliminates a number of very interesting examples.

(1.10.2) One could simply ask for a spectrum with a multiplication map that is only commutative, unital, and associative up to homotopy. In effect, these are *multiplicative cohomology theories*. This, by contrast, is too weak: there is no way to describe a well-behaved ∞-category of such objects, and there is no good homotopy theory of modules over such objects. The point here is that the homotopy that measures the failure of commutativity or associativity should itself be subject to coherence conditions.

We can think about the second of this failing strategies in the following way. Denote by $\mathbf{F}$ the category of finite sets. This is a symmetric monoidal category under disjoint union. For any ordinary symmetric monoidal category $\mathscr{A}$, the category of monoid objects in $\mathscr{A}$ can be described as the category of symmetric monoidal functors $\mathbf{F} \longrightarrow \mathscr{A}$. Of course for an ordinary symmetric monoidal category, this is very wasteful: one doesn't need any of the sets with more than 3 elements to express the associativity and commutativity constraints. However, if we replace A with a symmetric monoidal ∞-category, this is no longer true: if n is a positive integer, a symmetric monoidal functor $\mathbf{F} \longrightarrow \mathscr{A}$ is not determined by its value on sets of cardinality $\leq n$.

Definition 1.11. — The ∞-category of *E_∞ ring spectra* — or, more briefly, *E_∞ rings* — is the ∞-category of symmetric monoidal functors $\mathbf{F} \longrightarrow \mathscr{S}p$.

Definition 1.12. — A still different point of view on E_∞ ring spectra can be obtained through the homotopy theory of *operads*. It would take us a little far afield to describe this theory fully here, but it will suffice to simply point out that operads parametrize algebraic structures. For any operad $\mathscr{O}$, there is a notion of $\mathscr{O}$-algebra. Associative rings, commutative rings, Lie algebras, Poisson algebras, etc., etc., can all be described as algebras over an operad.

Now as we observed, strictly commutative rings in homotopy theory are not reasonable objects. Another way of saying this is to say that the commutative operad is *not cofibrant*. Hence one has to take a kind of "injective resolution" (i.e., a *cofibrant replacement*) of the commutative operad. The result is then called an *E_∞ operad*. Such a cofibrant replacement can actually be constructed topologically, by looking at configuration spaces of points in a high-dimensional affine space.

1.13. — If E is a spectrum with a multiplication map that is only commutative, unital, and associative up to homotopy (i.e., a multiplicative cohomology theory), then the homotopy groups $\pi_* E$ form a graded-commutative ring.

If, however, E is an E_∞ ring, there is a far richer structure available: for any prime p, there are natural homomorphisms $Q^s : \pi_*(H\mathbf{F}_p \wedge E) \longrightarrow \pi_{*+k}(H\mathbf{F}_p \wedge E)$, where $k = 2s(p-1)$ if p is odd, and $k = s$ if $p = 2$. These operations enjoy the following properties.

(1.13.1) If p is odd and $2s < |x|$, or if $p = 2$ and $s < |x|$, then $Q^s(x) = 0$.
(1.13.2) If p is odd and $2s = |x|$, or if $p = 2$ and $s = |x|$, then $Q^s(x) = x^p$.
(1.13.3) $Q^s(1) = 0$, where $1 \in \pi_0(H\mathbf{F}_p \wedge E)$ is the multiplicative unit.

(1.13.4) The *Cartan formula* holds:
$$Q^s(xy) = \sum_{i+j=s} Q^i(x)Q^{j(y)}.$$
(1.13.5) The *Adem relations* hold: if $r > ps$, then
$$Q^r Q^s = \sum_i (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r} Q^{r+s-i} Q^i,$$
and if p is odd and $r \geq ps$, then
$$Q^r \beta Q^s = \sum_i (-1)^{r+i} \binom{(p-1)(i-s)}{pi-r} \beta Q^{r+s-i} Q^i - \sum_i (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r-1} Q^{r+s-i} \beta Q^i,$$
where β is the Bockstein.

(1.13.6) The *Nishida relations* hold: for n sufficiently large,
$$P^r Q^s = \sum_i (-1)^{r+i} \binom{p^n+(p-1)(s-r)}{r-pi} Q^{s-r+i} P^i,$$
where P^r is the dual Steenrod power operation; for p odd and n sufficiently large,
$$P^r \beta Q^s = \sum_i (-1)^{r+i} \binom{p^n+(p-1)(s-r)-1}{r-pi} \beta Q^{s-r+i} P^i - \sum_i (-1)^{r+i} \binom{p^n+(p-1)(s-r)-1}{r-pi-1} Q^{s-r+i} P^i \beta.$$

In fact, this is only the beginning. These operations actually exist on any H_∞ *ring spectrum* — a still weaker notion than E_∞ ring spectrum! One sees that the structure of an E_∞ ring is very rich, and very intricate.

As a result, it can be extremely difficult to find E_∞ structures on a fixed spectrum. At the moment, there are two standard obstruction theories for determining whether a given multiplicative cohomology theory comes from an E_∞ ring spectrum, and each of them is well beyond our scope here.

For our purposes, it is most convenient for us to work with E_∞ rings that arise naturally *as E_∞ rings* — not merely as spectra on which we will have to find an E_∞ structure.

Example 1.14. — (1.14.1) For any commutative ring R, the Eilenberg-Mac Lane spectrum HR is an E_∞ ring.

(1.14.2) Since the sphere spectrum $\mathbf{S}$ is the unit for the smash product, it is an E_∞ ring. It is moreover the initial E_∞ ring, just as $\mathbf{Z}$ is the initial commutative ring. Observe, however, that whereas $\mathbf{Z}$ is a (somewhat) simple algebraic object, the sphere spectrum is hair-raisingly complex.

(1.14.3) The complex K-theory spectrum KU is an E_∞ ring spectrum. The resulting graded-commutative ring structure is
$$\pi_* KU \cong \mathbf{Z}[u^{\pm}],$$
where $|u| = 2$. One may form the *connective cover* of KU, obtaining a conective E_∞ ring ku, with
$$\pi_* KU \cong \mathbf{Z}[u],$$
where $|u| = 2$.

(1.14.4) For any E_∞ ring A, one may contemplate the *free E_∞ A-algebra* $A[x]$ on one generator. As we have seen, free objects are not easy to understand in homotopy theory, and this is no exception. In particular, even if A is an Eilenberg–Mac Lane spectrum, $R[x]$ is not an Eilenberg–Mac Lane spectrum. For example,
$$\pi_j((H\mathbf{Z})[x]) \cong \bigoplus_{m \geq 0} H_j(\Sigma_m, \mathbf{Z}).$$
Observe in particular that the natural morphism $(H\mathbf{Z})[x] \longrightarrow H(\mathbf{Z}[x])$ is very very far from being an equivalence.

1.15. — So E_∞ rings will provide the affines for *topological algebraic geoemtry*. More precisely, we will let $\mathscr{A}\!\mathit{ff}$ be the opposite ∞-category to the category of E_∞ rings. The objects thereof will be called *affines*, and we will denote by $\operatorname{Spec} A$ the object corresponding to an E_∞ ring A. If X is a fixed affine, then $\mathscr{A}\!\mathit{ff}_{/X}$ will denote the ∞-category of affines over X.

We will denote by $\mathscr{Aff}_{\geq 0}$ the opposite category to the category of *connective* E_∞ rings. The objects thereof will be called *connective affines*. If X is a fixed conenctive affine, then $\mathscr{Aff}_{\geq 0,/X}$ will denote the ∞-category of affines over X.

Kinds of modules. — Since E_∞ rings have good homotopy theories of modules attached to them, we can study some of the geometry of E_∞ rings by studying their ∞-categories of modules.

Definition 1.16. — For any E_∞ ring A, the ∞-category $\mathscr{Mod}(A)$ of *A-modules* is the ∞-category of $\mathscr{Sp}$-enriched functors $A \longrightarrow \mathscr{Sp}$. Using the symmetric monoidal structure on $\mathscr{Sp}$ combined with the symmetric monoidal structure on A coming from the E_∞ structure, one may conclude that $\mathscr{Mod}(A)$ inherits a symmetric monoidal structure $-\wedge_A -$.

Definition 1.17. — Suppose A an E_∞ ring, and suppose M an A-module.

(1.17.1) Suppose n an integer; then M has Tor*-amplitude* $\leq n$ if for every discrete A-module N (i.e., for every A-module N such that $\pi_j N = 0$ if $j \neq 0$), the tensor product $M \otimes_A N$ is n-truncated (i.e., $\pi_j(M \otimes_A N) = 0$ if $j > n$).

(1.17.2) We say that M is *perfect* or *compact* if for any filtered diagram $N : \Lambda \longrightarrow \mathscr{Mod}(A)$, one has

$$\mathbf{R}\,\underline{\mathrm{Mor}}(M, \mathrm{hocolim}_{\alpha\in\Lambda} N_\alpha) \simeq \mathrm{hocolim}_{\alpha\in\Lambda} \mathbf{R}\,\underline{\mathrm{Mor}}(M, N_\alpha).$$

(1.17.3) We say that M is *pseudocoherent* if there exists a simplicial A-module $M_\bullet$ such that $M \simeq \mathrm{hocolim}\, M_\bullet$, and for any nonnegative integer $n \geq 0$, M_n is a retract of a finite wedge of suspensions of A.

Proposition 1.18. — *If A is a noetherian E_∞ ring — i.e., if A is connective and $\pi_0 A$ is noetherian as an ordinary ring — then an A-module M is pseudocoherent if and only if both of the following conditions are satisfied.*

(1.18.1) M is m-connective for some integer m.

(1.18.2) $\pi_j M$ is finitely generated as a $\pi_0 A$-module for every $j \in \mathbf{Z}$.

Definition 1.19. — If A is an E_∞ ring and $M_\bullet$ is a simplicial A-module $M_\bullet$, there is a spectral sequence

$$E^2_{n,*} := \pi_n \pi_* M_\bullet \Longrightarrow \pi_{n+*}\, \mathrm{hocolim}\, M_\bullet.$$

We say that $M_\bullet$ is *E^r cohomologically bounded* if there exists an integer c such that the $E^r_{n,*}$ term of this spectral sequence is concentrated in a vertical band $a \leq n \leq c$. We say that $M_\bullet$ is *cohomologically bounded* if it is E^r cohomologically bounded for some integer $r > 1$.

We say that a pseudocoherent A-module M is *coherent* if the simplicial A-module $M_\bullet$ as in (1.17.3) can be be chosen to be cohomologically bounded.

Theorem 1.20. — *Suppose A an E_∞ ring. Then an A-module M is perfect if and only if it is coherent and of finite* Tor*-amplitude.*

Definition 1.21. — An E_∞ ring is *regular* if every coherent module is perfect.

Example 1.22. — (1.22.1) If R is an ordinary regular ring, then HR is a regular E_∞ ring.

(1.22.2) The E_∞ ring *ku* is regular.

(1.22.3) The sphere spectrum $\mathbf{S}$ is *not* regular. Again we see that free objects are not so simple in homotopy theory.

The Zariski site of $\mathrm{Spec}\, A$**. —** From the point of view of the Zariski site, the affine $\mathrm{Spec}\, A$ corresponding to a connective E_∞ ring A is $\mathrm{Spec}\, \pi_0 A$ along with a sheaf of E_∞ algebras.

Definition 1.23. — (1.23.1) A morphism $f : Y = \mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A = X$ is a *Zariski open immersion* if the induced functor

$$f_\star : \mathscr{Mod}(Y) \longrightarrow \mathscr{Mod}(X)$$

is fully faithful.

(1.23.2) A collection $\{f_\alpha : Y_\alpha = \operatorname{Spec} B_\alpha \longrightarrow \operatorname{Spec} A = X\}_{\alpha\in A}$ is a *formal covering* if the functor

$$\prod_{\alpha\in A} f_\alpha : \mathscr{P}erf(X) \longrightarrow \prod_{\alpha\in A} \mathscr{P}erf(Y_\alpha)$$

is conservative.

(1.23.3) A *Zariski covering* of an affine $\operatorname{Spec} A$ is a formal covering $\{f_\alpha : Y_\alpha = \operatorname{Spec} B_\alpha \longrightarrow \operatorname{Spec} A = X\}_{\alpha\in A}$ such that each f_α is a Zariski open immersion.

1.24. — One can show that the Zariski coverings generate a topology on the ∞-category of affines (giving the *large Zariski site*) and on the ∞-category of Zariski open immersions into a fixed affine $\operatorname{Spec} A$ (giving the *small Zariski site*).

Proposition 1.25. — *For a morphism $f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ of connective affines, the following are equivalent.*

(1.25.1) The induced morphism $\operatorname{Spec} \pi_0 B \longrightarrow \operatorname{Spec} \pi_0 A$ is a Zariski open immersion.

(1.25.2) The morphism f is a Zariski open immersion.

1.26. — Suppose A an E_∞ ring, and suppose M an A-module. An A-module N is said to be *M-acyclic* if $M \wedge_A N \simeq 0$; an A-module P is said to be *M-local* if for any M-acyclic A-module N, $[N,P] = 0$. There is a construction, due to Bousfield, of a left adjoint

$$L_M : \mathscr{M}od(A) \longrightarrow L_M\, \mathscr{M}od(A)$$

to the inclusion $L_M\, \mathscr{M}od(A) \subset \mathscr{M}od(A)$ of the full subcategory spanned by the M-local A-modules. This left adjoint is called a *Bousfield localization*.

One says that the Bousfield localization L_M is *smashing* if the natural morphism $L_M \longrightarrow - \wedge_A L_M A$ is an equivalence. In this case, $L_M A$ is an E_∞ ring with an E_∞ ring map $A \longrightarrow L_M A$, and the ∞-category $L_M\, \mathscr{M}od(A)$ can be identified with $\mathscr{M}od(L_M A)$. Hence the morphism

$$\operatorname{Spec} L_M A \longrightarrow \operatorname{Spec} A$$

is a Zariski open immersion. One can very that all Zariski open immersions arise this way.

Flatness. — Another important notion of topological algebraic geometry is the analogue of the notion of *flat families*.

Theorem 1.27. — *The following are equivalent for a morphism $f : Y = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = X$ of connective affines.*

(1.27.1) As an A-module, B can be written as a filtered colimit of finitely generated free A-modules.

(1.27.2) For any discrete A-module M, the B-module $f^\star M := M \otimes_A B$ is discrete.

(1.27.3) The functor

$$f^\star : \mathscr{M}od(X) \longrightarrow \mathscr{M}od(Y)$$

is left t-exact, so that it carries $\mathscr{M}od(X)_{\leq 0}$ into $\mathscr{M}od(Y)_{\leq 0}$.

(1.27.4) The following pair of conditions is satisfied.

(1.27.4.1) The induced homomorphism $\operatorname{Spec} \pi_0 B \longrightarrow \operatorname{Spec} \pi_0 A$ is a flat morphism of ordinary schemes.

(1.27.4.2) For every integer $j \in \mathbf{Z}$, the homomorphism

$$\pi_j A \otimes_{\pi_0 A} \pi_0 B \longrightarrow \pi_j B$$

of $\pi_0 B$-modules is an isomorphism.

In this case, the morphism f will be called flat.

Theorem 1.28. — *The following are equivalent for a flat morphism $f : Y = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = X$ of connective affines.*

(1.28.1) The functor

$$f^\star : \mathscr{M}od(X) \longrightarrow \mathscr{M}od(Y)$$

is conservative, so that for any nonzero A-module M, the B-module $f^\star M := M \otimes_A B$ is nonzero.

(1.28.2) The induced morphism $\operatorname{Spec} \pi_0 B \longrightarrow \operatorname{Spec} \pi_0 A$ is faithfully flat.

In this case, the morphism f will be called faithfully flat.

1.29. — A simplicial object $V_\bullet$ of $\mathscr{Aff}_{\geq 0,/U}$ is a *flat hypercovering* of an affine U if for any integer $n \geq 0$, the morphism

$$(\mathrm{sk}_{n-1} V_\bullet)_n \longrightarrow U$$

is faithfully flat.

Along with Čech nerves of covering families

$$\{ U_i \longrightarrow \coprod_{j\in I} U_j \}_{i\in I},$$

the flat hypercoverings generate the *flat hypertopology* $\flat$ on the ∞-category $\mathscr{Aff}_{\geq 0,/X}$ for an affine X. The corresponding *flat ∞-topos* $\mathscr{S}^\flat(\mathscr{Aff}_{\geq 0,/X})$ is hypercomplete. A presheaf

$$F : \mathscr{Aff}^{\mathrm{op}}_{\geq 0,/X} \longrightarrow \mathscr{Kan}$$

is a *flat hypersheaf over X* if it lies in $\mathscr{S}^\flat(\mathscr{Aff}_{\geq 0,/X})$, i.e., if the following two conditions are satisfied.

(1.29.1) For any object $U \in \mathscr{Aff}_{\geq 0,/X}$ and any flat hypercovering $V_\bullet$ of U, the induced morphism

$$F U \longrightarrow \lim F V_\bullet$$

is an equivalence.

(1.29.2) For any object $U = \coprod_{i\in I} U_i \in \mathscr{Aff}_{\geq 0,/X}$, the induced morphism

$$F U \longrightarrow \prod_{i\in I} F U_i$$

is an equivalence.

Theorem 1.30. — *Suppose X an affine. The flat hypertopology $\mathscr{Aff}_{\geq 0,/X}$ is subcanonical. Moreover, the assignments*

$$\mathscr{Mod} : U \longmapsto \mathscr{Mod}(U) \qquad \textit{and} \qquad \mathscr{Perf} : U \longmapsto \mathscr{Perf}(U)$$

are flat hypersheaves of ∞-categories. In particular, the associated presheaves $\iota\mathscr{Mod}$ and $\iota\mathscr{Perf}$ of spaces are flat hypersheaves.

The cotangent complex; smooth and étale. — One of the easiest pieces of classical algebraic geometry to transfer to the derived setting is Illusie's *cotangent complex*.

1.31. — Suppose $f : Y = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = X$ a morphism of affines. For any B-module M, one has an associated *square zero extension $B \oplus M$* of B. Write $Y_M := \operatorname{Spec}(B \oplus M)$; there is an obvious morphism $Y \longrightarrow Y_M$. Now define the *space of derivations on Y over X with coefficient in M* as the fiber $\mathrm{Der}_X(Y; M)$ of the morphism of spaces

$$\mathrm{Mor}_X(Y_M, Y) \longrightarrow \mathrm{Mor}_X(Y, Y)$$

over the identity map. The result is a functor

$$\mathrm{Der}_X(Y; -) : \mathscr{Mod}(T) \longrightarrow \mathscr{S}.$$

Theorem 1.32. — *For any morphism $f : Y = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = X$ of affines, the functor $\mathrm{Der}_X(Y;-)$ is corepresentable; that is, there exists a B-module $\mathbf{L}_{Y|X}$ and an equivalence*

$$\mathrm{Der}_X(Y; M) \simeq \mathrm{Mor}(\mathbf{L}_{Y|X}, M),$$

functorial in M. The representing object $\mathbf{L}_{Y|X}$ is called the cotangent complex for f, *and the morphism*

$$d : Y_{\mathbf{L}_{Y|X}} \longrightarrow Y$$

corresponding to the identity of $\mathbf{L}_{Y|X}$ is called the universal derivation for f.

1.33. — If R and S are (discrete) $\mathbf{Q}$-algebras, and if $R \longrightarrow S$ is a morphism thereof, then it can be shown that the cotangent complex $\mathbf{L}_{S|R}$ of Illusie coincides with $\mathbf{L}_{\operatorname{Spec} HS|\operatorname{Spec} HR}$. If, however, R and S are not $\mathbf{Q}$-algebras, this fails dramatically.

For example, let us consider the stalk of $\mathbf{L}_{H(\mathbf{F}_p[x])|H\mathbf{F}_p}$ at the origin 0; it is a result of Birgit Richter that

$$\mathbf{L}_{(H(\mathbf{F}_p[x])|H\mathbf{F}_p),0} \simeq H\mathbf{Z} \wedge H\mathbf{F}_p.$$

Once again one sees the dramatic difference between $H(R[x])$ and $(HR)[x]$ for commutative rings R.

1.34. — Suppose $f : Y = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = X$ a morphism of affines, locally of finite presentation (so that B is a compact object in the category of E_∞ rings under A).

(1.34.1) One says that f is *smooth* if $\mathbf{L}_{Y|X}$ is compact in $\mathscr{M}od(X)$.

(1.34.2) One says that f is *étale* if $\mathbf{L}_{Y|X} \simeq 0$.

Theorem 1.35. — *The following are equivalent for a morphism* $f : Y = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = X$ *of affines.*

(1.35.1) The morphism f *is étale.*

(1.35.2) The induced morphism on topological Hochschild homology spectra

$$\mathrm{THH}(A) \longrightarrow \mathrm{THH}(B)$$

is an equivalence.

(1.35.3) The following pair of conditions is satisfied.

(1.35.3.1) The induced homomorphism $\operatorname{Spec} \pi_0 B \longrightarrow \operatorname{Spec} \pi_0 A$ *is an étale morphism of ordinary schemes.*

(1.35.3.2) For every integer $j \in \mathbf{Z}$*, the homomorphism*

$$\pi_j A \otimes_{\pi_0 A} \pi_0 B \longrightarrow \pi_j B$$

of $\pi_0 B$*-modules is an isomorphism.*

1.36. — Suppose X an affine. A family

$$\{V_i \longrightarrow U\}_{i \in I}$$

is an *étale covering* if each morphism $V_i \longrightarrow U$ is étale, and for some finite subset $I' \subset I$, the morphism

$$\coprod_{i \in I'} V_i \longrightarrow U$$

is faithfully flat.

These families generate the *étale topology* ét on the ∞-category $\mathscr{A}\!f\!f_{\geq 0,/X}$. The corresponding *étale ∞-topos* $\mathscr{S}^{\text{ét}}(\mathscr{A}\!f\!f_{\geq 0,/X})$ is not hypercomplete.

2. Complex cobordism as a torsor

Thom spectra. — The Eilenberg–Mac Lane functor offered us a way of turning ordinary affines into affines in topological algebraic geometry. There are two more "topological" approaches to constructing affines in topological algebraic geometry, which we'll introduce here.

2.1. — Recall that a pointed space X is said to be an *infinite loop space* if there is a spectrum E and an equivalence

$$X \simeq \Omega^\infty E.$$

These data comprise an *infinite loop space structure* on X.

A theorem of Peter May provides a recognition principle for infinite loop space structures on a pointed space X. It can be expressed (somewhat informally) in the following manner. For any E_∞ operad $\mathscr{O}$, a $\mathscr{O}$-algebra structure on X such that the induced monoid $\pi_0 X$ is a group is *the same thing as* an infinite loop space structure on X. This is May's celebrated *infinite loop space machine*.

Put differently, an infinite loop space structure on a pointed space X is tantamount to the structure of an abelian group "up to coherent homotopy" on X. It is perhaps not to surprising that one can form the "group ring" spectrum

$$\mathbf{S}[X] := \Sigma^\infty X_+,$$

which inherits an E_∞ ring structure from the infinite loop spaces structure on X.

But there's more: the diagonal map $X \longrightarrow X \times X$ induces a coproduct

$$\mathbf{S}[X] \longrightarrow \mathbf{S}[X] \wedge \mathbf{S}[X].$$

This coproduct is coassociative and counital up to coherent homotopy; thus, one may regard $\mathbf{S}[X]$ as a commutative E_∞ Hopf algebra. Consequently,

$$\mathbf{G}(X) := \operatorname{Spec} \mathbf{S}[X]$$

is a *proalgebraic group*!

2.2. — The adjunction between spaces and spectra

$$\Sigma^\infty_+ : \mathscr{S} \rightleftarrows \mathscr{S}p : \Omega^\infty$$

can be lifted to an adjunction between E_∞ spaces (i.e., $\mathscr{O}$-algebras for any E_∞ operad $\mathscr{O}$) and E_∞ rings:

$$\Sigma^\infty_+ : E_\infty(\mathscr{S}) \rightleftarrows E_\infty(\mathscr{S}p) : \Omega^\infty.$$

That is, if A is an E_∞ ring, then $\Omega^\infty E$ is an E_∞ space in a canonical fashion. In particular, $\pi_0 \Omega^\infty A$ is an abelian monoid; thus one can throw away all the components of $\Omega^\infty A$ that correspond to noninvertible elements of $\pi_0 \Omega^\infty A$ by forming the pullback

$$\begin{array}{ccc} \mathrm{GL}_1(A) & \longrightarrow & \Omega^\infty A \\ \downarrow & & \downarrow \\ (\pi_0 \Omega^\infty A)^\times & \longrightarrow & \pi_0 \Omega^\infty A. \end{array}$$

The result is a pair of adjunctions

$$\{\text{Infinite loop spaces}\} \underset{\text{invertibles}}{\overset{\text{May}}{\rightleftarrows}} E_\infty(\mathscr{S}) \underset{\Omega^\infty}{\overset{\Sigma^\infty_+}{\rightleftarrows}} E_\infty(\mathscr{S}p),$$

and the composition of the two right adjoints is the functor GL_1.

Now one can form $B\,\mathrm{GL}_1\,\mathbf{S}$, which is the classifying space for spherical fibrations; that is, for any finite CW complex X and any spherical fibration $E \longrightarrow X$, there is a unique map

$$\zeta : X \longrightarrow B\,\mathrm{GL}_1\,\mathbf{S}$$

so that E is the pullback of $E\,\mathrm{GL}_1\,\mathbf{S}$ along ζ. The *Thom spectrum* X^ζ is the spectrification of the prespectrum whose spaces are Thom spaces of spherical fibrations:

$$\mathrm{Th}(X,\zeta)(n) := \mathrm{Th}(X, \zeta \wedge S(e^n))$$

with structure maps given by the obvious maps

$$\Sigma\,\mathrm{Th}(X, \zeta \wedge S(e^n)) \longrightarrow \mathrm{Th}(X, \zeta \wedge S(e^{n+1})).$$

The following remarkable theorem describes the homotopy groups of the associated spectrum.

***Theorem 2.3* (Pontryagin–Thom Construction).** — *Suppose (to fix ideas) ξ is a vector bundle on X of rank k. Then*

$$\pi_{n+k}(X^{S(\xi)})$$

is the set of equivalence classes of triples (M, g, ϕ) — where M is a closed manifold of dimension n, $g : M \longrightarrow X$ is a map, and ϕ is an isomorphism of bundles $N_{M|X} \cong g^\star \xi$ — under the relation of bordism.

***Theorem 2.4* (Lewis).** — *If the map $\zeta : X \longrightarrow B\,\mathrm{GL}_1\,\mathbf{S}$ is an n-fold loop map, then the Thom spectrum X^ζ inherits a canonical E_n ring spectrum structure. In particular, if ζ is an infinite loop map, then X^ζ is an E_∞ ring.*

2.5. — Thus if $\zeta : X \longrightarrow B\,\mathrm{GL}_1\,\mathbf{S}$ is an infinite loop map, then we have an affine

$$\mathbf{M}^\zeta(X) := \operatorname{Spec} X^\zeta.$$

The Thom isomorphism and $\mathbf{G}(X)$-torsors. — The Thom isomorphism is a classical result of topology which has a remarkable meaning in the context of topological algebraic geometry.

***Theorem 2.6* (Thom Isomorphism).** — *Suppose $\zeta : X \longrightarrow B\,\mathrm{GL}_1\,\mathbf{S}$ is an infinite loop map. Then there is a canonical morphism*

$$X^{\zeta} \longrightarrow X^{\zeta} \wedge X_{+},$$

called the Thom diagonal *which induces an* equivalence

$$X^{\zeta} \wedge X^{\zeta} \longrightarrow X^{\zeta} \wedge X_{+}.$$

2.7. — In geometric terms, the Thom diagonal is an action

$$\mathbf{G}(X) \times_{\operatorname{Spec}\mathbf{S}} \mathbf{M}^{\zeta}(X) \longrightarrow \mathbf{M}^{\zeta}(X)$$

of the proalgebraic group $\mathbf{G}(X)$ on $\mathbf{M}^{\zeta}(X)$ such that the induced "shear" morphism

$$\mathbf{G}(X) \times_{\operatorname{Spec}\mathbf{S}} \mathbf{M}^{\zeta}(X) \longrightarrow \mathbf{M}^{\zeta}(X) \times_{\operatorname{Spec}\mathbf{S}} \mathbf{M}^{\zeta}(X)$$

is an equivalence of affines.

Definition 2.8. — For any morphisms $A \longrightarrow B$ of E_∞ rings, the *Amitsur complex* is the cosimplicial A-algebra

$$\begin{aligned} \mathscr{A}^{\bullet}(B|A) :\ \Delta &\longrightarrow (A/E_\infty(\mathscr{S}p)) \\ \mathbf{p} &\longmapsto B \wedge_A B \wedge_A \cdots \wedge_A B. \end{aligned}$$

This is a simplicial affine $\operatorname{Spec}\mathscr{A}^{\bullet}(B|A)$.

There's an obvious augmentation

$$A \longrightarrow \operatorname{holim}\mathscr{A}^{\bullet}(B|A).$$

If it is an equivalence, then $\operatorname{Spec} A$ is said to be *complete along* $\operatorname{Spec} B$.

***Theorem 2.9* (Adams, Novikov).** — $\operatorname{Spec}\mathbf{S}$ *is complete along* $\mathbf{M}^{\zeta}(X)$ *for any infinite loop map* $\zeta : X \longrightarrow B\,\mathrm{GL}_1\,\mathbf{S}$.

Corollary 2.10. — *For any infinite loop map $\zeta : X \longrightarrow B\,\mathrm{GL}_1\,\mathbf{S}$, the Thom diagonal gives $\mathbf{M}^{\zeta}(X)$ the structure of a $\mathbf{G}(X)$-torsor over* $\operatorname{Spec}\mathbf{S}$.

Adams–Novikov spectral sequence. — The torsor structure of $\mathbf{M}^{\zeta}(X)$ for an infinite loop map $\zeta : X \longrightarrow B\,\mathrm{GL}_1\,\mathbf{S}$ provides an incredibly powerful tool for computing homotopy groups of the sphere spectrum.

Corollary 2.11. — *The Bousfield–Kan spectral sequence to compute $\pi_*\operatorname{holim}\mathscr{A}^{\bullet}(X^{\zeta}|\mathbf{S})$ gives the* Adams–Novikov spectral sequence

$$E_2^{s,t} := \operatorname{Ext}_{X_*^{\zeta}(X)}(X_*^{\zeta}, X_*^{\zeta}) \Longrightarrow \pi_{t-s}\mathbf{S}.$$

Example 2.12. — If we take ζ to be the spherical fibration given by $S(EU)$, where EU is the universal U-bundle over BU, then we get the *complex cobordism* spectrum MU. A computation of Milnor and Novikov shows that

$$\pi_* MU \cong \mathbf{Z}[x_1, x_2, \ldots],$$

where $x_j \in \pi_{2j} MU$.

Now $\operatorname{Spec} MU$ is, as we have seen, a $\mathbf{G}(BU)$-torsor over $\operatorname{Spec}\mathbf{S}$. The Adams–Novikov spectral sequence in this case takes a familiar form:

$$E_2^{s,t} := \operatorname{Ext}_{MU_*(BU)}(MU_*, MU_*) \Longrightarrow \pi_{t-s}\mathbf{S}.$$

Despite the challenge of computing the E_2 term effectively, this tool has been instrumental in the low-level computations of $\pi_*\mathbf{S}$.

3. The chromatic tower in stable homotopy theory

Johnson–Wilson spectra and Morava K-theories. — Let us now work p-locally for a prime p (which we will suppress in the notation). The Adams–Novikov spectral sequence exhibits fascinating structures on $\pi_*\mathbf{S}$, called v_n *periodicity*.

3.1. — In the Adams–Novikov spectral sequence, the most interesting contributions from

$$\pi_* MU \cong \mathbf{Z}_{(p)}[x_1, x_2, \dots]$$

are from special elements

$$v_n = x_{p^n-1} \in \pi_{2(p^n-1)} MU.$$

By convention, we think of v_0 as p.

The reason for this is a tad beyond our scope here, but let it suffice to say that the pair $(\pi_* MU, \pi_*(MU \wedge MU))$ is an affine groupoid which is isomorphic to the moduli stack of (smooth, 1-dimensional) formal group laws. This is true even integrally. When we move to the p-local world, it is convenient to concern ourselves with p-typical formal group laws; this has the effect of killing all the generators in $\pi_* MU$ apart from the v_n's. The result is a fascinating spectrum BP with homotopy

$$\pi_* BP \cong \mathbf{Z}_{(p)}[v_1, v_2, \dots].$$

It is not known whether BP is E_∞, so it is an interesting challenge to find a algebro-geometric interpretation of this object.

3.2. — There's much more to this story. Given a formal group law F over a graded ring A_* classified by a morphism of graded rings $\pi_* MU \longrightarrow A_*$, one can ask when the functor

$$X \longmapsto A_* \otimes_{\pi_* MU} MU_*(X)$$

is a homology theory. The Landweber Exact Functor Theorem gives a condition under which this is true. These are the so-called *Landweber exact homology theories*. A whole host of spectra arise from this construction, but we will be particularly interested in two of these.

(3.1.1) Johnson–Wilson spectra $E(n)$. These are obtained from BP by killing all the v_j's above degree n and inverting n:

$$\pi_* E(n) \cong \mathbf{Z}_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}].$$

By convention, we set $E(0) = H\mathbf{Q}$.

(3.1.2) Morava K-theories $K(n)$. These are much smaller; they are obtained from $E(n)$ by killing p and all the generators except v_n:

$$\pi_* K(n) \cong \mathbf{F}_p[v_n, v_n^{-1}].$$

We will use these spectra to obtain a lot of geometric information about $\operatorname{Spec}\mathbf{S}$.

Chromatic convergence and a covering of $\operatorname{Spec}\mathbf{S}$. — The spectra $E(n)$ and $K(n)$ described above can be used to paint a fascinating picture of $\operatorname{Spec}\mathbf{S}$.

3.3. — We saw above that Zariski opens of an affine $\operatorname{Spec} A$ are in bijection with the *smashing Bousfield localizations*

$$L_M : \mathscr{M}od(A) \longrightarrow L_M \mathscr{M}od(A).$$

As it happens, when $M = E(n)$, the Johnson–Wilson spectrum, the Bousfield localization $L_{E(n)}$ on modules over $\mathbf{S}$ is smashing. Hence we have a sequence

$$\mathbf{S} \longrightarrow \cdots \longrightarrow L_{E(n)}\mathbf{S} \longrightarrow L_{E(n-1)}\mathbf{S} \longrightarrow \cdots \longrightarrow L_{E(1)}\mathbf{S} \longrightarrow L_{E(0)}\mathbf{S} \simeq H\mathbf{Q}.$$

of E_∞ rings that correspond to a nested sequence

$$\operatorname{Spec} H\mathbf{Q} \simeq \operatorname{Spec} L_{E(0)}\mathbf{S} \longrightarrow \operatorname{Spec} L_{E(1)}\mathbf{S} \longrightarrow \cdots \longrightarrow \operatorname{Spec} L_{E(n-1)}\mathbf{S} \longrightarrow \operatorname{Spec} L_{E(n)}\mathbf{S} \longrightarrow \cdots \longrightarrow \operatorname{Spec}\mathbf{S}$$

of Zariski open immersions.

Theorem 3.4* (Chromatic Convergence, Hopkins–Ravenel).** — *Suppose M a perfect* **S-module. Then the homotopy limit*

$$\operatorname{holim}[\cdots \longrightarrow L_{E(n)}M \longrightarrow L_{E(n-1)}M \longrightarrow \cdots \longrightarrow L_{E(1)}M \longrightarrow L_{E(0)}M]$$

is M, and in fact

$$\pi_* M \cong \lim_{n\to\infty} \pi_* L_{E(n)}M.$$

3.5. — This remarkable theorem shows that our nested sequence of Zariski open immersions

$$\operatorname{Spec} H\mathbf{Q} \simeq \operatorname{Spec} L_{E(0)}\mathbf{S} \longrightarrow \operatorname{Spec} L_{E(1)}\mathbf{S} \longrightarrow \cdots \longrightarrow \operatorname{Spec} L_{E(n-1)}\mathbf{S} \longrightarrow \operatorname{Spec} L_{E(n)}\mathbf{S} \longrightarrow \cdots \longrightarrow \operatorname{Spec} \mathbf{S}$$

actually comprises a *Zariski open cover*. This is the *chromatic filtration* on $\operatorname{Spec}\mathbf{S}$ (or, equivalently, on the ∞-category $\mathcal{S}p$).

This leads to a fascinating picture of $\operatorname{Spec}\mathbf{S}$: whereas the Zariski topology of $\operatorname{Spec}\mathbf{Z}$ is generated by open sets obtained by inverting a single prime, in $\operatorname{Spec}\mathbf{S}$, we see that at each prime p, there is an infinite family of localizations that together generate the Zariski topology of $\operatorname{Spec}\mathbf{S}$. This leads to the idea of a *chromatic prime*, which is essentially a pair (p,n) consisting of prime p and a *level* n.

One word of warning, however: even though the limit $\lim_n \pi_* L_{E(n)}\mathbf{S}$ is π_*, and hence zero in negative degrees, the E_∞ rings $L_{E(n)}\mathbf{S}$ are highly nonconnective. So we do not regard $\operatorname{Spec} L_{E(n)}\mathbf{S}$ as a nil-thickenings of $\operatorname{Spec} \pi_0 L_{E(n)}\mathbf{S}$; the geometry of $\operatorname{Spec} L_{E(n)}\mathbf{S}$ is far subtler.

The monochromatic layer. — If we wish to understand the geometry of $\operatorname{Spec}\mathbf{S}$ inductively, we are led to understand the difference between $\operatorname{Spec} L_{E(n)}\mathbf{S}$ and $\operatorname{Spec} L_{E(n-1)}\mathbf{S}$. This is where the Morava K-theories $K(n)$ introduced above become useful.

3.6. — In effect, the complement of $\operatorname{Spec} L_{E(n-1)}\mathbf{S}$ in $\operatorname{Spec} L_{E(n)}\mathbf{S}$ is a closed "subscheme" of $\operatorname{Spec}\mathbf{S}$, and we wish to study its formal thickening. It turns out that this is effectively modeled by the *monochromatic sphere spectrum* $L_{K(n)}\mathbf{S}$.

3.7. — It is important to note that $L_{K(n)}$ is not a smashing localization, so we do not wish to contemplate the ∞-category $\mathcal{M}od(L_{K(n)}\mathbf{S})$ itself. Rather, it is better to think of the ∞-category $L_{K(n)}\mathcal{S}p$ of *monochromatic spectra* as a starting point for *$K(n)$-local algebraic geometry*. Indeed, the functor $L_{K(n)}(-\wedge -)$ defines a symmetric monoidal structure on $L_{K(n)}\mathcal{S}p$, for which $L_{K(n)}\mathbf{S}$ is of course the unit. This symmetric monoidal structure is a kind of *completed tensor product*.

The upshot of this kind of thinking is that one should contemplate not $\operatorname{Spec} L_{K(n)}\mathbf{S}$, but rather the *formal scheme* $\operatorname{Spf} L_{K(n)}\mathbf{S}$. We regard the category $L_{K(n)}\mathcal{S}p$ as the symmetric monoidal ∞-category of modules over $\operatorname{Spf} L_{K(n)}\mathbf{S}$. The E_∞ rings in $L_{K(n)}\mathcal{S}p$ can be interpreted geometrically as formal affines over $\operatorname{Spf} L_{K(n)}\mathbf{S}$.

The idea that $\operatorname{Spf} L_{K(n)}\mathbf{S}$ is a formal thickening of the complement of $\operatorname{Spec} L_{E(n-1)}\mathbf{S}$ in $\operatorname{Spec} L_{E(n)}\mathbf{S}$ is expressed by the following square, which is both a pushout and a pullback of E_∞ rings:

$$\begin{tikzcd} L_{E(n)}\mathbf{S} \arrow[r] \arrow[d] & L_{E(n-1)}\mathbf{S} \arrow[d] \\ L_{K(n)}\mathbf{S} \arrow[r] & L_{E(n-1)}L_{K(n)}\mathbf{S}. \end{tikzcd}$$

We think of this square as dual to a gluing square

$$\begin{tikzcd} \operatorname{Spf} L_{E(n-1)}L_{K(n)}\mathbf{S} \arrow[r] \arrow[d] & \operatorname{Spf} L_{K(n)}\mathbf{S} \arrow[d] \\ \operatorname{Spec} L_{E(n-1)}\mathbf{S} \arrow[r] & \operatorname{Spec} L_{E(n)}\mathbf{S}. \end{tikzcd}$$

Once again, the geometry of $\operatorname{Spf} L_{K(n)}\mathbf{S}$ is complicated by the fact that it is highly nonconnective.

Lubin–Tate spectra. — Hopkins and Miller introduced a sequence of $K(n)$-local spectra with a remarkable property: they are Galois extensions of the monochromatic sphere.

3.8. — The *Lubin–Tate spectrum* E_n is a $K(n)$-local Landweber exact homology theory whose associated formal group is the Lubin–Tate universal deformation of the Honda formal group H_n of height n over $\mathbf{F}_{p^n}$ (with p-series $[p]_n(x) = x^{p^n}$), so that

$$\pi_* E_n \cong \mathbf{W}_{\mathbf{F}_{p^n}}[[u_1, \dots, u_{n-1}]][u^{\pm}],$$

where each $u_i \in \pi_0 E_n$, and $u \in \pi_2 E_n$; in particular, E_n is even periodic. By a theorem of Goerss–Hopkins–Miller, there is a canonical E_∞ ring structure on E_n. Hence we are led to contemplate $\mathrm{Spf}\, E_n$, a formal affine over $\mathrm{Spf}\, L_{K(n)}\mathbf{S}$.

Theorem 3.9 **(Devinatz–Goerss–Hopkins–Miller–Rognes).** — *The morphism* $\mathrm{Spf}\, E_n \longrightarrow \mathrm{Spf}\, L_{K(n)}\mathbf{S}$ *is an étale cover, and its group of automorphisms (in the category of $K(n)$-local formal schemes) is the* nth extended Morava stabilizer group

$$\mathbf{G}_n := \mathrm{Gal}(\mathbf{F}_{p^n} | \mathbf{F}_p) \ltimes \mathrm{Aut}\, H_n,$$

a p-adic Lie group. There is an associated spectral sequence

$$E^2_{s,t} := H^{-s}(\mathbf{G}_n; \pi_t E_n) \Longrightarrow \pi_{s+t} L_{K(n)}\mathbf{S}.$$

3.10. — Geometrically, this suggests a lot of promise. In particular, one is led to think of $\mathrm{Spf}\, L_{K(n)}\mathbf{S}$ as a kind of quotient stack $[\mathrm{Spf}\, E_n / \mathbf{G}_n]$. In particular, one is particularly interested in the *adèlic sphere spectrum*:

$$\mathbf{A_S} := H\mathbf{Q} \vee \bigvee_{n>0} L_{K(n)}\mathbf{S},$$

whose associated geometric object can be thought of as a coproduct

$$\mathrm{Spec}\, H\mathbf{Q} \sqcup \coprod_{n>0} [\mathrm{Spf}\, E_n / \mathbf{G}_n].$$

Example 3.11. — When $n = 1$, $L_{K(1)}\mathbf{S}$ is the p-adic image-of-J spectrum, E_1 is the p-adic K-theory spectrum $KU^\wedge_p$, and the Morava stabilizer group is $\mathbf{Z}^\times_p$, which acts on $KU^\wedge_p$ via Adams operations $k \longmapsto \psi^k$. The connective cover e_1 of E_1 is $ku^\wedge_p$,

When $n = 2$, Shimomura and Wang have computed $L_{E(2)}\mathbf{S}$ when $p = 3$. I do not understand it.

4. Toward $K(\mathbf{S})$

Why compute $K(\mathbf{S})$?— As we will see, computing the algebraic K-theory of the sphere spectrum is a daunting task. It is therefore reasonable that we might offer some motivation for doing so.

4.1. — First, and most importantly, recall that the algebraic K-theory of number rings contains remarkable arithmetic information. For example, recall that the *Lichtenbaum Conjecture* states that, for any number field F, the special value of the zeta function of F at positive integers is controlled by the size of K-theory groups:

$$|\zeta^\star_F(1-m)| \underset{(2)}{=} \frac{\#^\tau K_{2m-2}(\mathcal{O}_F)}{\#^\tau K_{2m-1}(\mathcal{O}_F)} R^B_{F,m}$$

(where the notation $\underset{(2)}{=}$ indicates that one has equality up to powers of 2), where $R^B_{F,m}$ is the *Borel regulator*, defined as the covolume of the image lattice of the *Borel regulator map*

$$\rho^B_{F,m} : K_{2m-1}(\mathcal{O}_F) \longrightarrow \mathbf{R}^{d_m}.$$

A key objective of topological algebraic geometry is to extract similar arithmetic information from the K-theory of the sphere spectrum.

4.2. — There is also tremendous practical value in computing the algebraic K-theory of various E_∞ ring spectra. To illustrate this, suppose now X a pointed CW complex; then Waldhausen's algebraic K-theory spectrum $A(X)$ is canonically equivalent to the algebraic K-theory $K(\mathbf{S}[\Omega X])$. Moreover, for any pointed space X, there is a split fiber sequence

$$Q(X_+) \longrightarrow \Omega^\infty K(\mathbf{S}[\Omega X]) \longrightarrow \mathrm{Wh}(X),$$

where $A(X)$ is the algebraic K-theory of X, and $\mathrm{Wh}(X)$ is the smooth Whitehead space. If X is a compact manifold, Waldhausen's stable parametrized h-cobordism theorem expresses a remarkable equivalence between $\Omega\mathrm{Wh}(X)$ and the space of h-cobordisms on $X \times I^{\times N}$ for N sufficiently large. Furthermore, results of Farrell–Jones indicate that if X is a Riemannian manifold with nonpositive sectional curvature (say), then $\mathrm{Wh}(X)$ can be assembled from $\mathrm{Wh}(\{x\})$ for every point $x \in X$ and $\mathrm{Wh}(\gamma)$ for every closed geodesic $\gamma \subset X$. Putting all this together, if X is a closed Riemannian manifold with nonpositive sectional curvature, then the space of h-cobordisms on $X \times I^{\times N}$ for N sufficiently large is the loop space of a summand of a space constructed from $Q(X_+)$, $K(\mathbf{S})$, and $K(\mathbf{S}[\mathbf{Z}])$.

4.3. — So how do we compute $K(\mathbf{S})$? One complicating factor is that the sphere spectrum is not regular; hence its K-theory and its G-theory do not coincide, and some results (such as dévissage) fail.

Comparing $K(\mathbf{S})$ and $K(\mathbf{Z})$. — The point of view of the Zariski site is that $\operatorname{Spec}\mathbf{S}$ is a kind of topological nil-thickening of $\operatorname{Spec}\mathbf{Z}$. Hence the K-theories of $\mathbf{S}$ and $\mathbf{Z}$ should be closely linked. Indeed, the map $\mathbf{S} \to H\mathbf{Z}$ induces an equivalence $K(\mathbf{S}, H\mathbf{Q}) \to K(\mathbf{Z}, H\mathbf{Q})$, and hence by Borel:

$$K_n(\mathbf{S}, H\mathbf{Q}) = K_n(\mathbf{Z}, H\mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{if } n = 0; \\ \mathbf{Q} & \text{if } n = 4k+1,\ k > 0; \\ 0 & \text{else.} \end{cases}$$

Moreover, after p-completion there is the Dundas–McCarthy homotopy pullback square:

$$\begin{tikzcd} K(\mathbf{S})^\wedge_p \arrow[r] \arrow[d] & K(\mathbf{Z})^\wedge_p \arrow[d] \\ TC(\mathbf{S})^\wedge_p \arrow[r] & TC(\mathbf{Z})^\wedge_p \end{tikzcd}$$

where the vertical maps are the p-complete cyclotomic trace maps. Unfortunately, the description of $K(\mathbf{S})$ provided by this relationship is still computationally very intricate.

Using the chromatic filtration. — A point of view advertised by Waldhausen and Rognes proposes to organize the homotopy into periodic families by describing $K(\mathbf{S})$ via the chromatic tower, *viz.*:

$$\cdots \longrightarrow K(L_{E(n)}\mathbf{S}) \longrightarrow K(L_{E(n-1)}\mathbf{S}) \longrightarrow \cdots \longrightarrow K(L_{E(1)}\mathbf{S}) \longrightarrow K(L_{E(0)}\mathbf{S}) \simeq K(\mathbf{Q}).$$

One is thus led in particular to attempt to compute the algebraic K-theory $K(\operatorname{Spf} L_{K(n)}\mathbf{S})$ of the monochromatic spheres — or even of the adèlic sphere spectrum.

Following Hopkins, Waldhausen, and others, Ausoni and Rognes proposed the following conjectures of Beilinson–Lichtenbaum type:

Conjecture 4.4. — *Suppose V an $\mathbf{S}$-module of chromatic type $(n+1)$, and suppose T the mapping telescope of its v_{n+1}-self-map. Then the natural morphism*

$$K(\operatorname{Spf} L_{K(n)}\mathbf{S}) \wedge T \to K(\operatorname{Spf} E_n)^{h\mathbf{G}_n} \wedge T$$

is an equivalence.

***Conjecture 4.5* (Chromatic Red-shift).** — *With V and T as above, the natural morphism*

$$K(\operatorname{Spf} L_{K(n)}\mathbf{S}) \wedge V \longrightarrow K(\operatorname{Spf} L_{K(n)}\mathbf{S}) \wedge T$$

is an equivalence in sufficiently high degrees. This gives rise to a Beilinson-Lichtenbaum spectral sequence

$$E^2_{s,t} := H^{-s}(\mathbf{G}_n; K_t(\operatorname{Spf} E_n, V)) \Longrightarrow K_{s+t}(\operatorname{Spf} L_{K(n)}\mathbf{S}, V),$$

which converges for $s+t \gg 0$.

Example 4.6. — When $n=0$, the algebraic K-theory (with suitable coefficients) of $e_0 = H\mathbf{Z}_p$ was computed by Bökstedt–Madsen and Rognes. When $n=1$, Rognes asked whether there is a localization sequence:

$$K\mathbf{Z}_p \longrightarrow K(ku_p^\wedge) \longrightarrow K(KU_p^\wedge).$$

Such a localization sequence was established by Blumberg–Mandell. Thus for the case $n=1$, it remains to compute $K(ku_p^\wedge)$. Ausoni–Rognes have done this modulo p and v_1, and the answer is compatible with conjecture 4.4.

4.7. — Following ideas of Gunnar Carlsson, I have developed a kind of refinement to conjecture 4.4 using *equivariant topological algebraic geometry*.

Consider the ∞-category $A_n := \mathscr{P}erf_{L_{K(n)}\mathbf{S}}(B\mathbf{G}_n)$ of finite $K(n)$-local representations of the Morava stabilizer group $\mathbf{G}_n$; this is naturally a *$K(n)$-local tannakian ∞-category*. As a result, the algebraic K-theory of A_n naturally admits a structure of a $\mathbf{G}_n$-equivariant E_∞ ring spectrum, which I denote $\mathbf{K}(A_n)$.

Since $\mathbf{G}_n$ acts on $\operatorname{Spf} E_n$, one may also form a $\mathbf{G}_n$-equivariant E_∞ ring spectrum $\mathbf{K}(A_n;\operatorname{Spf} E_n)$. There is a canonical morphism

$$\alpha : \mathbf{K}(A_n) \longrightarrow \mathbf{K}(A_n;\operatorname{Spf} E_n).$$

4.8. — In the $\mathbf{G}_n$-equivariant setting, complete information about Green functors can be obtained by considering the values on finite $\mathbf{G}_n$-sets, in particular on finite orbits $(\mathbf{G}_n/H)$. We have the following observations.

(4.8.1) The Green functor $\pi_*\mathbf{K}(A_n)$ assigns to any orbit $(\mathbf{G}_n/H)$ the K-theory of the ∞-category $\mathscr{P}erf_{\operatorname{Spf} L_{K(n)}\mathbf{S}}(BH)$ of $K(n)$-local representations of H. In particular,

$$\pi_*^{\{1\}}\mathbf{K}(A_n) \cong K_*(\operatorname{Spf} L_{K(n)}\mathbf{S}) \qquad \text{and} \qquad \pi_*^{\mathbf{G}_n}\mathbf{K}(A_n) \cong K_*\mathscr{P}erf_{\operatorname{Spf} L_{K(n)}\mathbf{S}}(B\mathbf{G}_n).$$

(Strictly speaking, of course, the subgroup $\{1\} \subset G_k$ isn't actually an option here; rather, I want to suggest that $\mathbf{K}(A_n)$ is an attempt at interpolation between $K(\operatorname{Spf} L_{K(n)}\mathbf{S})$ and $K\mathscr{P}erf_{\operatorname{Spf} L_{K(n)}\mathbf{S}}(B\mathbf{G}_n)$.)

(4.8.2) The Green functor $\pi_*\mathbf{K}(A_n;\operatorname{Spf} E_n)$ assigns to any orbit $(\mathbf{G}_n/H)$ the K-theory of $\operatorname{Spf}(E_n^{hH})$. In particular,

$$\pi_*^{\{1\}}\mathbf{K}(A_n;\operatorname{Spf} E_n) \cong K_*(\operatorname{Spf} E_n) \qquad \text{and} \qquad \pi_*^{\mathbf{G}_n}\mathbf{K}(A_n;\operatorname{Spf} E_n) \cong K_*(\operatorname{Spf} L_{K(n)}\mathbf{S}).$$

Observe that α is very from being an equivalence.

4.9. — Both $\mathbf{K}(A_n)$ and $\mathbf{K}(A_n;\operatorname{Spf} E_n)$ admit a "monochromatic dimension" morphism to the constant Green functor at the monochromatic sphere $L_{K(n+1)}\mathbf{S}$, and the following diagram commutes:

$$\begin{tikzcd} \mathbf{K}(A_n) \arrow[rr] \arrow[dr] & & \mathbf{K}(A_n;\operatorname{Spf} E_n) \arrow[dl] \\ & L_{K(n+1)}\mathbf{S}. & \end{tikzcd}$$

One may regard $\operatorname{Spf} L_{K(n+1)}\mathbf{S}$ as a closed subscheme in both $\operatorname{Spec}\mathbf{K}(A_n;\operatorname{Spf} E_n)$ and $\operatorname{Spec}\mathbf{K}(A_n)$, and one may think of α as a morphism of equivariant affines that respects this closed subscheme. One may further form the formal completions of these schemes along this subscheme, and study the behavior of α on these.

Conjecture 4.10. — *The morphism on the completions*

$$\mathbf{K}(A_n)^\wedge_{K(n+1)} \longrightarrow \mathbf{K}(A_n;\operatorname{Spf} E_n)^\wedge_{K(n+1)}.$$

is an equivalence.

Conjecture 4.11. — *The value of $\mathbf{K}(A_n;\operatorname{Spf} E_n)^\wedge_{K(n+1)}$ on $(\mathbf{G}_n/\mathbf{G}_n)$ agrees with $L_{K(n+1)}K(\operatorname{Spf} L_{K(n)}\mathbf{S})$.*

Clark Barwick, Department of Mathematics, Harvard University, One Oxford Street, Cambridge, MA 02138, USA
E-mail : `clarkbar@gmail.com`