THE FUNDAMENTAL GROUPOID AND THE POSTNIKOV TOWER

18.904

1. Numerically generated spaces

Let us agree now that the word *space* means "topological space," and the word *map* means "continuous map." If we wish to speak of an ordinary mapping between sets, with no continuity demands, we will use the phrase *set map*.

1.1. **Definition.** For any space Y, a *test map* is a map $V \longrightarrow Y$, where V is an open subset of some Euclidean space $\mathbf{R}^N$.

Suppose X a (topological) space. A subset $U \subset X$ is *numerically open* if for any test map $\phi \colon V \longrightarrow X$, the inverse image $\phi^{-1}(U) \subset V$ is open.

1.2. **Lemma.** *Any open set of a space is numerically open; however, there exist spaces that contain numerically open sets that are not open.*

1.3. **Definition.** We will say that a space X is *numerically generated* if every numerically open set is open.

1.4. **Example.** *Any open subset of a Euclidean space* $\mathbf{R}^N$ *is numerically generated.*

1.5. **Lemma.** *Any open subset of a numerically generated space is numerically generated.*

1.6. **Lemma.** *Suppose* X *and* Y *numerically generated spaces. Then a function* $X \longrightarrow Y$ *is continuous just in case, for any test map* $V \longrightarrow X$*, the composite* $V \longrightarrow Y$ *is continuous.*

1.7. **Proposition.** *The disjoint union of any family of numerically generated spaces is numerically generated.*

1.8. **Notation.** Let us write $I := [0, 1] \subset \mathbf{R}$.

1.9. **Proposition.** *The following are equivalent for a space* X.

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(1.9.1) X *is numerically generated.*
(1.9.2) A subset $U \subset X$ *is open if, for any map* $\phi\colon I \longrightarrow X$*, the inverse image* $\phi^{-1}(U) \subset I$ *is open.*
(1.9.3) A subset $U \subset X$ *is open if, for any map* $\phi\colon \mathbf{R} \longrightarrow X$*, the inverse image* $\phi^{-1}(U) \subset \mathbf{R}$ *is open.*

1.10. **Example.** *The poorly named "topologist's sine curve"*

$$\{(x,y) \in \mathbf{R}^2 \mid [x \neq 0] \wedge [y = \sin(1/x)]\} \cup \{(0,0)\} \subset \mathbf{R}^2$$

is not numerically generated.

1.11. **Proposition.** *The collection of numerically open subsets of a space* X *form a new topology that is as fine as the original topology on* X.

1.12. **Definition.** Suppose X a space. The set X equipped with the topology on a space X given by the previous proposition will be called the *numericalization* of X, and it will be denoted $X^\sharp$. (So an open set of $X^\sharp$ is precisely a numerically open set of X.)

1.13. **Proposition.** *For any space* X*, the space* $X^\sharp$ *is numerically generated. Furthermore, the identity on* X *is a map* $j_X\colon X^\sharp \longrightarrow X$ *with the following property: for any numerically generated space* T *and any map* $g\colon T \longrightarrow X$*, there exists a unique map* $g^\sharp\colon T \longrightarrow X^\sharp$ *such that the triangle*

$$\begin{array}{ccccc} & & X^\sharp & & \\ & {}^{g^\sharp}\nearrow & & \searrow^{j} & \\ T & & \xrightarrow[g]{\hspace{3em}} & & X. \end{array}$$

commutes.

1.13.1. **Corollary.** *For any space* X*, one has*

$$(X^\sharp)^\sharp = X^\sharp.$$

1.13.2. **Corollary.** *For any map* $g\colon X \longrightarrow Y$*, there exists a unique map*

$$g^\sharp\colon X^\sharp \longrightarrow Y^\sharp$$

such that following diagram commutes:

$$\begin{array}{ccc} X^\sharp & \xrightarrow{g^\sharp} & Y^\sharp \\ {\scriptstyle j_X}\downarrow & & \downarrow{\scriptstyle j_Y} \\ X & \xrightarrow[g]{} & Y. \end{array}$$

1.14. **Example.** *Consider* $\mathbf{Q} \subset \mathbf{R}$ *with its subspace topology. Then* $\mathbf{Q}$ *is not numerically generated, as* $\mathbf{Q}^\sharp$ *is discrete.*

1.15. **Definition.** Suppose that X, Y, and Z are three sets, and suppose that $p\colon X \longrightarrow Z$ and $q\colon Y \longrightarrow Z$ are two maps of sets. Then the subset

$$X \times_Z Y := \{(x, y) \in X \times Y \mid p(x) = q(y)\} \subset X \times Y$$

is called the *fiber product of* X *and* Y *over* Z. (When Z is the one-point space $*$, of course $X \times_Z Y = X \times Y$.)

Suppose X, Y, and Z numerically generated spaces, and suppose that p and q are continuous. If we endow $X \times Y$ with the product topology, then we can equip $X \times_Z Y$ with the subspace topology. However, we will go one step further and consider the numericalization of these topologies. We will just denote the resulting numerically generated spaces as

$$X \times_Z Y \subset X \times Y$$

(without any further decoration). We will call this the *numerically generated fiber product* of X and Y over Z.

1.16. **Notation.** For any spaces X and Y, write Map(X, Y) for the set of maps $X \longrightarrow Y$.

1.17. **Proposition.** *Suppose that* X, Y, *and* Z *are numerically generated spaces, and suppose that* $p\colon X \longrightarrow Z$ *and* $q\colon Y \longrightarrow Z$ *are two maps. Then the numerically generated fiber product* $X \times_Z Y$ *enjoys the following universal property: for any numerically generated space* U*, the maps* $X \times_Z Y \longrightarrow X$ *and* $X \times_Z Y \longrightarrow Y$ *induce a bijection*

$$\mathrm{Map}(U, X \times_Z Y) \xrightarrow{\sim} \mathrm{Map}(U, X) \times_{\mathrm{Map}(U,Z)} \mathrm{Map}(U, Y).$$

1.18. **Definition.** Suppose X and Y two numerically generated spaces. For any compact subset $K \subset X$, and any open subset $W \subset Y$, write

$$U(K, V) := \{g \in \mathrm{Map}(X, Y) \mid \forall x \in K,\ g(x) \in W\}.$$

Then we may generate a topology on Map(X, Y) by the subbase consisting of all the sets U(K, W), called the *compact-open topology*. Again we will go one step further and consider the numericalization Map(X, Y) of this space. We will call this the *numerically generated mapping space* from X to Y.

1.19. **Proposition.** *Suppose that* X, Y, *and* Z *are numerically generated spaces. Then there is a natural homeomorphism*

$$\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

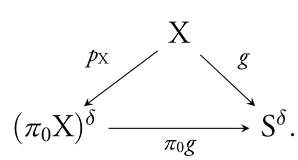
2. Existence and connectedness

2.1. **Notation.** For any set S, denote by S^δ the set S equipped with the discrete topology. Note that S^δ is numerically generated. For any set map $F\colon S \longrightarrow T$, we denote the corresponding map of spaces $S^\delta \longrightarrow T^\delta$ by F^δ.

2.2. **Definition.** Suppose X a space. Consider the equivalence relation $\sim$ on the points of X generated by declaring that $x \sim y$ if there exists a map $\gamma\colon I \longrightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Write $\pi_0 X$ for the set of equivalence classes of points of X under this equivalence relation. The elements of $\pi_0 X$ will be called *path components* of X. Write p_X for the set map $X \longrightarrow \pi_0 X$ that carries a point of X to its equivalence class.

2.3. **Example.** *For any set* S, *one has* $\pi_0(S^\delta) = S$. *Any Euclidean space* $\mathbf{R}^N$ *has* $\pi_0 \mathbf{R}^N = \{*\}$.

2.4. **Theorem.** *Suppose* X *a numerically generated space. Then the set map* p_X *is continuous as a map* $X \longrightarrow (\pi_0 X)^\delta$. *Furthermore, it has the following universal property: for any set* S *and any map* $g\colon X \longrightarrow S^\delta$, *there exists a unique set map* $\pi_0 g\colon \pi_0 X \longrightarrow S$ *such that the following diagram commutes:*



2.4.1. **Corollary.** *For any map* $g\colon X \longrightarrow Y$ *between numerically generated spaces, there exists a unique set map*

$$\pi_0 g\colon \pi_0 X \longrightarrow \pi_0 Y$$

such that following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ (\pi_0 X)^\delta & \xrightarrow[(\pi_0 g)^\delta]{} & (\pi_0 Y)^\delta. \end{array}$$

2.4.2. **Corollary.** *The following are equivalent for a numerically generated space* X.

(2.4.2.1) The set π_0X *consists of exactly one point.*
(2.4.2.2) There exists a point $x \in$ X *such that for any point* $y \in$ X*, there exists a map* $\gamma\colon$ I $\longrightarrow$ X *such that* $\gamma(0) = x$ *and* $\gamma(1) = y$.
(2.4.2.3) There is exactly one nonempty subset of X *that is both open and closed.*

2.5. **Example.** *The (still poorly named) "topologist's sine curve" of Ex. 1.10 satisfies condition (2.4.2.3) but not condition (2.4.2.2).*

2.6. **Definition.** A numerically generated space will be said to be *connected* if the equivalent conditions of Cor. 2.4.2 hold.

2.7. **Example.** *The empty space is not connected.*

2.8. **Proposition.** *Suppose* $g\colon$ X $\longrightarrow$ Y *a surjective map between numerically generated spaces. Then* Y *is connected if* X *is.*

2.9. **Example.** *For any natural number* $n \geq 1$*, the* n-sphere

$$\mathbf{S}^n := \{x \in \mathbf{R}^{n+1} \mid ||x|| = 1\}$$

is connected. However, $\mathbf{S}^0$ *is not connected.*

2.10. **Lemma.** *For any numerically generated space* X*, the set map*

$$\pi_0 \operatorname{id}_X\colon \pi_0 X \longrightarrow \pi_0 X$$

is the identity map.

2.11. **Proposition.** *Suppose that* X*,* Y*, and* Z *are numerically generated spaces, and suppose* $p\colon$ X $\longrightarrow$ Y *and* $q\colon$ Y $\longrightarrow$ Z *are two maps. Then the two set maps* π_0X $\longrightarrow \pi_0$Z *given by* $\pi_0(q \circ p)$ *and* $(\pi_0 q) \circ (\pi_0 p)$ *are equal.*

2.11.1. **Corollary.** *If* $g\colon$ X $\longrightarrow$ Y *is a homeomorphism between numerically generated spaces, then* $\pi_0 g\colon \pi_0$X $\longrightarrow \pi_0$Y *is a bijection.*

2.12. **Example.** *For any integer $n \neq 1$, the Euclidean spaces $\mathbf{R}$ and $\mathbf{R}^n$ are not homeomorphic.*

2.13. **Example.** *The capital letters* T *and* X *are not homeomorphic.*

2.14. **Example.** *For any integer $n \neq 1$, the Euclidean spaces $\mathbf{S}^1$ and $\mathbf{S}^n$ are not homeomorphic.*

2.15. **Proposition.** *For any numerically generated space* X *and any set* S*, the numerically generated space* $\mathrm{Map}(X, S^{\delta})$ *is discrete.*

2.15.1. **Corollary.** *For any numerically generated spaces* X *and* Y*, the two maps* $X \times Y \longrightarrow X$ *and* $X \times Y \longrightarrow Y$ *together induce a bijection*

$$\pi_0(X \times Y) \xrightarrow{\sim} \pi_0 X \times \pi_0 Y.$$

2.16. **Proposition.** *For any family* $\{X_i\}$ *of numerically generated spaces, the inclusions* $X_i \hookrightarrow \coprod_i X_i$ *together induce a bijection*

$$\coprod_i \pi_0(X_i) \cong \pi_0\left(\coprod_i X_i\right).$$

2.17. **Definition.** For any two numerically generated spaces X and Y, we will say that two maps $p, q \colon X \longrightarrow Y$ are *homotopic* if the images of p and q in $\pi_0 \mathrm{Map}(X, Y)$ are equal. In this case we write $p \simeq q$.

2.18. **Lemma.** *Two maps $p, q \colon X \longrightarrow Y$ are homotopic just in case there exists a map*

$$h \colon X \times I \longrightarrow Y$$

such that for any $x \in X$, one has

$$h(x, 0) = p(x) \qquad \textit{and} \qquad h(x, 1) = q(x).$$

2.19. **Definition.** We say that a map $\phi \colon X \longrightarrow Y$ between numerically generated spaces is a *homotopy equivalence* if there exists a map $\psi \colon Y \longrightarrow X$ such that both $\psi \circ \phi \simeq \mathrm{id}_X$ and $\phi \circ \psi \simeq \mathrm{id}_Y$.

2.20. **Proposition.** *A homotopy equivalence* $X \longrightarrow Y$ *between numerically generated spaces induces a bijection*

$$\pi_0 X \xrightarrow{\sim} \pi_0 Y.$$

3. Groupoids and groups

3.1. **Notation.** Suppose that X, Y, and Z are three sets, and suppose that $p\colon X \longrightarrow Z$ and $q\colon Y \longrightarrow Z$ are two set maps. Should we need to emphasize the role of the set maps p and q, we will denote the fiber product of X and Y over Z as

$$X \underset{p,Z,q}{\times} Y.$$

We will write

$$\mathrm{pr}_1 : X \underset{p,Z,q}{\times} Y \longrightarrow X$$

for the projection $(x, y) \longmapsto x$ and

$$\mathrm{pr}_2 : X \underset{p,Z,q}{\times} Y \longrightarrow Y$$

for the projection $(x, y) \longmapsto y$.

3.2. **Definition.** A *groupoid* $\Gamma = (M, O, s, t, i, c)$ consists of the following data:

(3.2.A) a set M, whose elements are called *isomorphisms* or *paths*,
(3.2.B) a set O, whose elements are called *objects*,
(3.2.C) two set maps $s, t\colon M \longrightarrow O$, which are called *source* and *target*, respectively,
(3.2.D) a set map $i\colon O \longrightarrow M$, called the *identity*, and
(3.2.E) a set map

$$c\colon M \underset{s,O,t}{\times} M \longrightarrow M,$$

called *composition*.

These data are subject to the following axioms.

(3.2.1) One has $s \circ i = t \circ i = \mathrm{id}$.
(3.2.2) One has

$$s \circ c = s \circ \mathrm{pr}_1 \quad \text{and} \quad t \circ c = t \circ \mathrm{pr}_2 .$$

(3.2.3) If $\phi \in M$, then

$$c(i(t(\phi)), \phi) = \phi \quad \text{and} \quad c(\phi, i(s(\phi))) = \phi.$$

(3.2.4) For any elements $\phi, \chi, \psi \in M$ such that $s(\phi) = t(\chi)$ and $s(\chi) = t(\psi)$, we have

$$c(\phi, c(\chi, \psi)) = c(c(\phi, \chi), \psi).$$

(3.2.5) For any element $\phi \in M$, there exists an element $\phi^{-1} \in M$ such that both

$$s(\phi) = t(\phi^{-1}) \quad \text{and} \quad t(\phi) = s(\phi^{-1}),$$

and both

$$c(\phi, \phi^{-1}) \quad \text{and} \quad c(\phi^{-1}, \phi)$$

are in the image of i.

3.3. **Notation.** In a groupoid $\Gamma = (M, O, s, t, i, c)$, if $\phi, \psi \in M$ are morphisms such that $s(\phi) = t(\psi)$, then we typically write

$$\phi \circ \psi := c(\phi, \psi).$$

Furthermore, for any two objects $x, y \in O$, we will denote by

$$\mathrm{Isom}_\Gamma(x, y)$$

for the fiber of the map $(s, t)\colon M \longrightarrow O \times O$ over the point (x, y). An element $\gamma \in \mathrm{Isom}_\Gamma(x, y)$ will typically be denoted

$$\gamma\colon x \xrightarrow{\sim} y.$$

3.4. **Lemma.** *A groupoid is precisely the same thing as a category in which every morphism is isomorphism.*

3.5. In general, when we specify a groupoid, we simply describe the objects, we describe the set of isomorphisms between any two objects, and, if necessary, we describe the composition.

3.6. **Example.** *For any set* S*, we obtain a groupoid* $S^\delta = (S, S, \mathrm{id}, \mathrm{id}, \mathrm{id}, \mathrm{id})$, *which we may call the* discrete *groupoid corresponding to* S.

3.7. **Example.** *We may consider the groupoid* Σ *of finite sets: the objects are finite sets, and an isomorphism*

$$S \xrightarrow{\sim} T$$

is simply a bijection.

3.8. **Example.** *If* k *is a field, we may consider* $\mathbf{Vect}(k)$, *the groupoid of finite dimensional vector spaces: the objects are finite dimensional vector spaces over* k*, and an isomorphism*

$$V \xrightarrow{\sim} W$$

is simply an isomorphism of k*-vector spaces.*

3.9. **Example.** *A group* G *gives rise to a groupoid (which we will also denote* G*)*

$$(G, *, !, !, e, c),$$

where $*$ *denotes a set with one element,* $!$ *denotes the unique map* $G \longrightarrow *$*, the map* $e\colon * \longrightarrow G$ *carries the unique element of* $*$ *to* $e \in G$*, and the map*

$$c\colon G \times G \longrightarrow G$$

is given by $c(g, h) = gh$*. So* $\mathrm{Isom}_G(*, *) \cong G$.

Every groupoid with exactly one object is of this form, so a group is *nothing more than a groupoid with exactly one object.*

3.10. **Example.** *Suppose* $\Gamma = (M, O, s, t, i, c)$ *and* $\Gamma' = (M', O', s', t', i', c')$ *two groupoids; then the product*

$$\Gamma \times \Gamma' = (M \times M', O \times O', s \times s', t \times t', i \times i', c \times c'),$$

is a groupoid.

3.11. **Definition.** If $\Gamma = (M, O, s, t, i, c)$ is a groupoid and $x \in O$ an object, then the composition law

$$M \underset{s,O,t}{\times} M \longrightarrow M$$

restricts to a group law $\mathrm{Isom}_\Gamma(x, x) \times \mathrm{Isom}_\Gamma(x, x) \longrightarrow \mathrm{Isom}_\Gamma(x, x)$. This group is called the *isotropy group* Γ_x of Γ at x.

3.12. **Example.** *Suppose* G *a group, and suppose* X *a* G*-set, i.e., a set with an action of* G *on the left. Write* α *for the action map* $G \times X \longrightarrow X$ *Then the* action groupoid *is the tuple*

$$G \ltimes X := (G \times X, X, \mathrm{pr}_2, \alpha, i, c),$$

where $i\colon X \longrightarrow G \times X$ *is simply the map* $x \longmapsto (e, x)$*, and the composition map*

$$c\colon (G \times X) \underset{\mathrm{pr}_2, X, \alpha}{\times} (G \times X) \longrightarrow G \times X$$

is given by the assignment $(g, hy, h, y) \longmapsto (gh, y)$*. So for any elements* $x, y \in X$*, we may identify*

$$\mathrm{Isom}_{G \ltimes X}(x, y) \cong \{g \in G \mid gx = y\}.$$

The isotropy group of $G \ltimes X$ *at a point* $x \in X$ *is the stabilizer of* x*.*

3.13. **Definition.** Suppose $\Gamma = (M, O, s, t, i, c)$ and $\Gamma' = (M', O', s', t', i', c')$ two groupoids; then a *morphism* $F\colon \Gamma' \longrightarrow \Gamma$ of groupoids is a pair of maps $F\colon M' \longrightarrow M$ and $F\colon O' \longrightarrow O$ such that

$$F \circ s' = s \circ F, \qquad F \circ t' = t \circ F, \qquad F \circ i' = i \circ F,$$

and, for any $\phi, \psi \in M$ with $s(\phi) = t(\psi)$, we have

$$F(c'(\phi, \psi)) = c(F(\phi), F(\psi)).$$

Composition of morphisms of groupoids is defined in the obvious manner, and a morphism $F\colon \Gamma' \longrightarrow \Gamma$ of groupoids is said to be an *isomorphism* if there exists a morphism $G\colon \Gamma \longrightarrow \Gamma'$ of groupoids such that $G \circ F = \mathrm{id}_{\Gamma'}$ and $F \circ G = \mathrm{id}_\Gamma$.

3.14. **Example.** *For any groupoid* Γ *and any object* x *thereof, the inclusion* $\Gamma_x \hookrightarrow \Gamma$ *is a morphism of groupoids.*

3.15. **Notation.** Suppose $\Gamma = (M, O, s, t, i, c)$ and $\Gamma' = (M', O', s', t', i', c')$ two groupoids. Then we may define a new groupoid $\mathrm{Mor}(\Gamma', \Gamma)$ as follows. The objects are morphisms of groupoids $\Gamma' \longrightarrow \Gamma$, and for two morphisms $F, G\colon \Gamma' \longrightarrow \Gamma$ of groupoids, let

$$\mathrm{Isom}_{\mathrm{Mor}(\Gamma',\Gamma)}(F, G) \subset \prod_{x \in O'} \mathrm{Isom}_{\Gamma}(Fx, Gx)$$

be the subset consisting of those tuples $(\eta_x\colon Fx \xrightarrow{\sim} Gx)_{x \in O'}$ such that for any isomorphism $\gamma\colon x \xrightarrow{\sim} y$ of Γ', one has

$$G(\gamma) \circ \eta_x = \eta_y \circ F(\gamma).$$

3.16. **Proposition.** *Suppose $\Gamma, \Gamma', \Gamma''$ three groupoids. Then there is a natural isomorphism of groupoids*

$$\mathrm{Mor}(\Gamma'' \times \Gamma', \Gamma) \cong \mathrm{Mor}(\Gamma'', \mathrm{Mor}(\Gamma', \Gamma)).$$

3.17. **Notation.** Write $\bar{I}$ for the groupoid that contains two objects 0 and 1 such that $\mathrm{Isom}_{\bar{I}}(x, y) = \{*\}$ for any $x, y \in \{0, 1\}$.

3.18. **Proposition.** *Suppose Γ and Γ' two groupoids, and suppose $F, G\colon \Gamma' \longrightarrow \Gamma$ two morphisms of groupoids. Then there is a natural bijection between*

$$\mathrm{Isom}_{\mathrm{Mor}(\Gamma',\Gamma)}(F, G)$$

and the set of morphisms of groupoids

$$H\colon \Gamma' \times \bar{I} \longrightarrow \Gamma$$

such that $H|(\Gamma' \times \{0\}^{\delta}) = F$ and $H|(\Gamma' \times \{1\}^{\delta}) = G$.

3.19. **Definition.** A morphism $F\colon \Gamma' \longrightarrow \Gamma$ of groupoids will be said to be an *equivalence of groupoids* if there exists a morphism $G\colon \Gamma \longrightarrow \Gamma'$ of groupoids such that both $\mathrm{Isom}_{\mathrm{Mor}(\Gamma',\Gamma')}(\mathrm{id}_{\Gamma'}, G \circ F)$ and $\mathrm{Isom}_{\mathrm{Mor}(\Gamma,\Gamma)}(\mathrm{id}_{\Gamma}, F \circ G)$ are nonempty. If such an equivalence exists, then Γ and Γ' are said to be *equivalent*.

3.20. **Definition.** Suppose $\Gamma = (M, O, s, t, i, c)$ a groupoid. Consider the equivalence relation $\sim$ on the objects of Γ given by declaring that $x \sim y$ just in case the set $\mathrm{Isom}_{\Gamma}(x, y)$ is nonempty. Write $\pi_0\Gamma$ for the set of equivalence classes of objects under this equivalence relation. The elements of Γ will be called *connected components* of Γ. Write p_{Γ} for the set map $O \longrightarrow \pi_0\Gamma$ that carries an object of Γ to its equivalence class.

3.21. **Example.** *For any set S, one has $\pi_0(S^{\delta}) = S$. Any group G has $\pi_0 G = \{*\}$.*

3.22. **Theorem.** *For any groupoid* Γ*, the set map* p_Γ *extends uniquely to a morphism of groupoids* $\Gamma \longrightarrow (\pi_0\Gamma)^\delta$*. Furthermore, it has the following universal property: for any set* S *and any morphism of groupoids* $F\colon \Gamma \longrightarrow S^\delta$*, there exists a unique set map* $\pi_0 F\colon \pi_0\Gamma \longrightarrow S$ *such that the following diagram of groupoids commutes:*

$$\begin{array}{ccc} & \Gamma & \\ {\scriptstyle p_\Gamma}\swarrow & & \searrow{\scriptstyle F} \\ (\pi_0\Gamma)^\delta & \xrightarrow[(\pi_0 F)^\delta]{} & S^\delta. \end{array}$$

3.22.1. **Corollary.** *For any morphism of groupoids* $F\colon \Gamma \longrightarrow \Gamma'$*, there exists a unique set map*

$$\pi_0 F\colon \pi_0\Gamma \longrightarrow \pi_0\Gamma'$$

such that following diagram commutes:

$$\begin{array}{ccc} \Gamma & \xrightarrow{F} & \Gamma' \\ {\scriptstyle p_\Gamma}\downarrow & & \downarrow{\scriptstyle p_{\Gamma'}} \\ (\pi_0\Gamma)^\delta & \xrightarrow[(\pi_0 F)^\delta]{} & (\pi_0\Gamma')^\delta. \end{array}$$

3.23. **Lemma.** *For any groupoid* Γ*, the set map*

$$\pi_0\, \mathrm{id}_\Gamma\colon \pi_0\Gamma \longrightarrow \pi_0\Gamma$$

is the identity map.

3.24. **Lemma.** *Suppose that* Γ, Γ'*, and* Γ'' *are groupoids, and suppose* $F\colon \Gamma' \longrightarrow \Gamma$ *and* $G\colon \Gamma'' \longrightarrow \Gamma'$ *are two maps. Then the two set maps* $\pi_0\Gamma'' \longrightarrow \pi_0\Gamma$ *given by* $\pi_0(F \circ G)$ *and* $(\pi_0 F) \circ (\pi_0 G)$ *are equal.*

3.25. **Proposition.** *An equivalence* $\Gamma' \longrightarrow \Gamma$ *between groupoids induces a bijection*

$$\pi_0\Gamma' \xrightarrow{\sim} \pi_0\Gamma.$$

3.25.1. **Corollary.** *The following are equivalent for a groupoid* Γ.

(3.25.1.1) The set $\pi_0\Gamma$ *consists of exactly one point.*

(3.25.1.2) There exists an object x *of* Γ *such that for any object* y *of* Γ*, the set* $\mathrm{Isom}_\Gamma(x, y)$ *is nonempty.*

(3.25.1.3) There exists a group G *and an equivalence of groupoids* $G \longrightarrow \Gamma$.

3.26. **Definition.** A groupoid will be said to be *connected* if the equivalent conditions of Cor. 3.25.1 hold.

3.27. **Proposition.** *A morphism* $F\colon \Gamma' \longrightarrow \Gamma$ *of groupoids is an equivalence if and only if the following two conditions obtain.*

(3.27.1) The morphism F *induces a bijection* $\pi_0\Gamma' \longrightarrow \pi_0\Gamma$.

(3.27.2) For any object $x \in \Gamma'$, *the induced homomorphism* $\Gamma'_x \longrightarrow \Gamma_{F(x)}$ *is an isomorphism.*

4. The Poincaré groupoid and the fundamental group

4.1. **Definition.** Suppose X a numerically generated space. Then a *path* in X from a point x to a point y is a map $\gamma\colon I \longrightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. The *space of paths* from x to y is the fiber $P_{x,y}(X)$ of the map

$$\mathrm{Map}(I, X) \longrightarrow X \times X$$

given by $\gamma \longmapsto (\gamma(0), \gamma(1))$ (As usual, we use the numerically generated fiber product.)

4.2. **Proposition.** *Suppose* X *a numerically generated space, and suppose* $x, y, z \in X$. *Then the map*

$$c_{x,y,z}\colon P_{y,z}(X) \times P_{x,y}(X) \longrightarrow P_{x,z}(X)$$

given by the formula

$$c_{x,y,z}(\beta, \alpha)(t) := \begin{cases} \alpha(2t) & \text{if } t \in [0, 1/2] \\ \beta(2t-1) & \text{if } t \in [1/2, 1] \end{cases}$$

is continuous.

4.3. **Proposition.** *For any map* $g\colon X \longrightarrow Y$ *of numerically generated spaces, and for any points* $x, y \in X$ *the map* $\mathrm{Map}(I, X) \longrightarrow \mathrm{Map}(I, Y)$ *restricts to a map*

$$g_\star\colon P_{x,y}(X) \longrightarrow P_{g(x),g(y)}(Y).$$

4.4. **Theorem.** *Suppose* X *a numerically generated space. Then there is a groupoid* $\Pi_1 X$ *whose objects are points of* X, *in which*

$$\mathrm{Isom}_{\Pi_1 X}(x, y) := \pi_0 P_{x,y}(X),$$

and composition is given by taking π_0 *of the map* $c_{x,y,z}$ *of the previous proposition:*

$$\pi_0 c_{x,y,z}\colon \pi_0 P_{y,z}(X) \times \pi_0 P_{x,y}(X) \cong \pi_0(P_{y,z}(X) \times P_{x,y}(X)) \longrightarrow \pi_0 P_{x,z}(X)$$

4.5. **Definition.** Suppose X a numerically generated space. The groupoid $\Pi_1(X)$ of the previous theorem is called the *fundamental groupoid* of the numerically generated space X. For any point $x \in X$, the isotropy group

$$\pi_1(X, x) := (\Pi_1 X)_x$$

is called the *fundamental group* of X.

4.6. **Proposition.** *For any map* $g\colon X \longrightarrow Y$ *of numerically generated spaces, the set maps*

$$g\colon X \longrightarrow Y \quad \textit{and} \quad \pi_0 g_\star\colon \pi_0 P_{x,y}(X) \longrightarrow \pi_0 P_{g(x),g(y)}(Y)$$

define a morphism of groupoids

$$\Pi_1 g\colon \Pi_1 X \longrightarrow \Pi_1 Y.$$

4.7. **Example.** *Consider the coproduct* $X := \mathbf{S}^1 \sqcup \mathbf{S}^1$, *and consider the action of* $\mathbf{Z}/2$ *on* X *obtained by switching the two summands. Then there is an induced action of* $\mathbf{Z}/2$ *on* $\Pi_1(X)$, *but for no point* $x \in X$ *is it the case that we obtain an induced action on* $\pi_1(X, x)$.

4.8. **Lemma.** *For any numerically generated space* X, *the set map*

$$\Pi_1\,\mathrm{id}_X\colon \Pi_1 X \longrightarrow \Pi_1 X$$

is the identity map.

4.9. **Proposition.** *Suppose that* X, Y, *and* Z *are numerically generated spaces, and suppose* $p\colon X \longrightarrow Y$ *and* $q\colon Y \longrightarrow Z$ *are two maps. Then the two set maps* $\Pi_1 X \longrightarrow \Pi_1 Z$ *given by* $\Pi_1(q \circ p)$ *and* $(\Pi_1 q) \circ (\Pi_1 p)$ *are equal.*

4.10. **Proposition.** *For any numerically generated space* X, *there is a natural bijection*

$$\pi_0 \Pi_1 X \cong \pi_0 X.$$

4.11. **Proposition.** *A homotopy equivalence* $g\colon X \xrightarrow{\sim} Y$ *induces an equivalence*

$$\Pi_1 g\colon \Pi_1 X \longrightarrow \Pi_1 Y$$

of groupoids.

4.12. **Example.** *If* $m \geq 1$, *the groupoid* $\Pi_1(\mathbf{R}^m)$ *is equivalent but not isomorphic to the trivial group.*

4.13. **Example.** *For any* $m \neq 2$, *the spaces* $\mathbf{R}^2$ *and* $\mathbf{R}^m$ *are not homeomorphic.*

4.14. **Example.** *For any* $m \geq 2$, *the map* $\mathbf{S}^m \longrightarrow *$ *induces an equivalence of fundamental groupoids.*

4.15. **Proposition.** *For any two numerically generated spaces* X *and* Y, *the two maps* $\Pi_1(\mathrm{pr}_1)\colon \Pi_1(X \times Y) \longrightarrow \Pi_1 X$ *and* $\Pi_1(\mathrm{pr}_2)\colon \Pi_1(X \times Y) \longrightarrow \Pi_1 Y$ *induce an isomorphism*

$$\Pi_1(X \times Y) \xrightarrow{\sim} \Pi_1 X \times \Pi_1 Y.$$

4.16. **Definition.** A *pointed space* (X, x) consists of a space X and a point (called the *basepoint*) $x \in X$. For any two pointed numerically generated spaces (X, x) and (Y, y), a pointed map is a map $g\colon X \longrightarrow Y$ such that $g(x) = y$. We write

$$\mathrm{Map}_*((X, x), (Y, y))$$

for the (numerically generated) fiber product

$$\mathrm{Map}(X, Y) \times_{\mathrm{Map}(\{x\}, Y)} \mathrm{Map}(\{x\}, \{y\}).$$

4.17. **Notation.** Consider the pointed space $(S^1, 1)$. For any pointed numerically generated space (X, x), write

$$\Omega_x X := \mathrm{Map}_*(S^1, X).$$

If the chosen point $x \in X$ is clear from the context, we may write ΩX for $\Omega_x X$.

Furthermore, we may regard ΩX as a pointed (numerically generated) space, where the basepoint is the constant map $c_x\colon S^1 \longrightarrow X$ at x. Consequently, we may iterate this construction to obtain, for every $n \geq 0$, a pointed space

$$\Omega^n X := \Omega\Omega^{n-1} X.$$

Now for any $n \geq 2$, write

$$\pi_n(X, x) := \pi_0 \Omega^n X.$$

4.18. **Proposition.** *For any pointed numerically generated space (X, x), there exists a natural isomorphism*

$$\pi_1(X, x) \cong \pi_0 \Omega_x X.$$

4.19. **Proposition.** *For any pointed numerically generated space (X, x), the group $\pi_1(\Omega_x X, c_x)$ is abelian.*

4.19.1. **Corollary.** *For any pointed numerically generated space (X, x) and for any $n \geq 2$, the group $\pi_n(X, x)$ is abelian.*

5. Sheaves and the étale fundamental groupoid

5.1. **Notation.** For any space X, write Op(X) for the following category. The objects are open sets of X, and a map $U \longrightarrow V$ is an inclusion $U \hookrightarrow V$; that is, there is a unique morphism $U \longrightarrow V$ if and only if $U \subset V$.

5.2. **Definition.** Suppose X a space. Then a *presheaf $\mathscr{F}$ on* X is a functor

$$\mathscr{F}\colon \mathrm{Op}(X)^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

For any open sets $U, V \in \mathrm{Op}(X)$, if $U \subset V$, we write $\rho_{U \subset V}$ for the set map $\mathscr{F}(V) \longrightarrow \mathscr{F}(U)$.

For an open set $\mathrm{U} \in \mathrm{Op}(\mathrm{X})$, an element $s \in \mathscr{F}(\mathrm{U})$ is sometimes called a *section of* $\mathscr{F}$ *over* U. An element of $\mathscr{F}(\mathrm{X})$ will be called a *global section*.

5.3. **Example.** *Suppose* X *and* Y *spaces. For any open set* $\mathrm{U} \in \mathrm{Op}(\mathrm{X})$*, write*

$$\mathscr{O}_{\mathrm{X}}^{\mathrm{Y}}(\mathrm{U})$$

for the set of maps $\mathrm{U} \longrightarrow \mathrm{Y}$*. This defines a presheaf* $\mathscr{O}_{\mathrm{X}}^{\mathrm{Y}}$ *on* X.

5.4. **Example.** *Suppose* $p\colon \mathrm{Y} \longrightarrow \mathrm{X}$ *a continuous map. Then for any open set* $\mathrm{U} \in \mathrm{Op}(\mathrm{X})$*, set*

$$\Gamma(p)(\mathrm{U}) = \Gamma(\mathrm{Y}/\mathrm{X})(\mathrm{U}) := \{s \in \mathscr{O}_{\mathrm{X}}^{\mathrm{Y}}(\mathrm{U}) \mid p \circ s = \mathrm{id}_{\mathrm{U}}\}.$$

We call $\Gamma(\mathrm{Y}/\mathrm{X})(\mathrm{U})$ *is the* set of sections of p over U*, and we call* $\Gamma(\mathrm{X}/\mathrm{Y})$ *the presheaf of local sections of* p*.*

5.5. **Proposition.** *Suppose* S *a set and suppose* X *a numerically generated space. For any open set* $\mathrm{U} \in \mathrm{Op}(\mathrm{X})$*, there is a natural bijection*

$$\mathrm{Map}(\pi_0 \mathrm{U}, \mathrm{S}) \cong \mathscr{O}_{\mathrm{X}}^{\mathrm{S}^{\delta}}(\mathrm{U}).$$

5.6. **Example.** *Write* $\mathbf{C}^{\times} := \mathbf{C} - \{0\}$*. Consider the map* $\mathrm{sq}\colon \mathbf{C}^{\times} \longrightarrow \mathbf{C}^{\times}$ *given by* $\xi \longmapsto \xi^2$*. Then the presheaf* $\Gamma(\mathrm{sq})$ *admits no global sections.*

5.7. **Example.** *Consider the exponential map* $\exp\colon \mathbf{C} \longrightarrow \mathbf{C}^{\times}$*. Then the presheaf* $\Gamma(\exp)$ *admits no global sections.*

5.8. **Example.** *For any set* S*, one may form the* constant presheaf $\mathscr{P}_{\mathrm{S}}$ *at* S*, which assigns to any open set* U *the set* S*, and to any open sets* $\mathrm{U}, \mathrm{V} \in \mathrm{Op}(\mathrm{X})$ *with* $\mathrm{V} \subset \mathrm{U}$ *the identity map on* S*.*

5.9. **Example.** *Suppose* X *a space, and suppose* $\mathrm{V} \in \mathrm{Op}(\mathrm{X})$ *a particular* fixed *open set. We have a presheaf* $\mathscr{H}_{\mathrm{V}}$ *defined by the rule*

$$\mathscr{H}_{\mathrm{V}}(\mathrm{U}) := \begin{cases} \{*\} & \textit{if} \quad \mathrm{U} \subset \mathrm{V} \\ \varnothing & \textit{otherwise.} \end{cases}$$

In this case, the restriction maps are unique: for any open sets $\mathrm{U}, \mathrm{U}' \in \mathrm{Op}(\mathrm{X})$ *with* $\mathrm{U}' \subset \mathrm{U}$*, there is a unique map* $\mathscr{H}_{\mathrm{V}}(\mathrm{U}) \longrightarrow \mathscr{H}_{\mathrm{V}}(\mathrm{U}')$.

The presheaf $\mathscr{H}_{\mathrm{V}}$ *is called the* presheaf represented by $\mathrm{V} \in \mathrm{Op}(\mathrm{X})$.

5.10. **Definition.** A *morphism* of presheaves $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ is a natural transformation. That is, ϕ consists of a tuple $(\phi_{\mathrm{U}})_{\mathrm{U} \in \mathrm{Op}(\mathrm{X})}$ of set maps

$$\phi_{\mathrm{U}}\colon \mathscr{F}(\mathrm{U}) \longrightarrow \mathscr{G}(\mathrm{U}),$$

subject to the following condition: for any open sets $\mathrm{U}, \mathrm{V} \in \mathrm{Op}(\mathrm{X})$ with $\mathrm{V} \subset \mathrm{U}$, the following diagram commutes:

$$\begin{array}{ccc} \mathscr{F}(\mathrm{U}) & \xrightarrow{\phi_{\mathrm{U}}} & \mathscr{G}(\mathrm{U}) \\ {\scriptstyle \rho_{\mathrm{V}\subset\mathrm{U}}}\downarrow & & \downarrow{\scriptstyle \rho_{\mathrm{V}\subset\mathrm{U}}} \\ \mathscr{F}(\mathrm{V}) & \xrightarrow[\phi_{\mathrm{V}}]{} & \mathscr{G}(\mathrm{V}). \end{array}$$

Write $\mathrm{Mor}_{\mathrm{X}}(\mathscr{F}, \mathscr{G})$ for the set of all morphisms of presheaves $\mathscr{F} \longrightarrow \mathscr{G}$.

Given morphisms of presheaves $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ and $\psi\colon \mathscr{G} \longrightarrow \mathscr{H}$, we can form the composite $\psi \circ \phi\colon \mathscr{F} \longrightarrow \mathscr{H}$ in the following manner: for any open set $\mathrm{U} \in \mathrm{Op}(\mathrm{X})$, set

$$(\psi \circ \phi)_{\mathrm{U}} := \psi_{\mathrm{U}} \circ \phi_{\mathrm{U}}.$$

This defines a morphism of presheaves $\mathscr{F} \longrightarrow \mathscr{H}$ as desired.

A morphism of presheaves $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ is said to be an *isomorphism* if there exists a morphism of sheaves $\psi\colon \mathscr{G} \longrightarrow \mathscr{F}$ such that both

$$\psi \circ \phi = \mathrm{id}_{\mathscr{F}} \quad \text{and} \quad \phi \circ \psi = \mathrm{id}_{\mathscr{G}} .$$

5.11. **Proposition.** *For any presheaf* $\mathscr{F}$ *on a space* X *and for any open set* $\mathrm{U} \in \mathrm{Op}(\mathrm{X})$, *there is a natural bijection*

$$\mathrm{Mor}_{\mathrm{X}}(\mathscr{H}_{\mathrm{U}}, \mathscr{F}) \cong \mathscr{F}(\mathrm{U}).$$

5.11.1. **Corollary.** *For any space* X *and any two open sets* $\mathrm{U}, \mathrm{V} \in \mathrm{Op}(\mathrm{X})$, *there is a morphism* $\iota\colon \mathscr{H}_{\mathrm{U}} \longrightarrow \mathscr{H}_{\mathrm{V}}$ *if and only if one has* $\mathrm{U} \subset \mathrm{V}$, *in which case* ι *is unique.*

5.12. **Example.** *For any two spaces* X *and* Y, *the presheaf* $\mathscr{O}_{\mathrm{X}}^{\mathrm{Y}}$ *is isomorphic to the presheaf of local sections* $\Gamma(\mathrm{Y} \times \mathrm{X}/\mathrm{X})$ *of the projection map* $\mathrm{pr}_2\colon \mathrm{Y} \times \mathrm{X} \longrightarrow \mathrm{X}$.

5.13. **Example.** *Suppose* S *is a set, and suppose* X *a space with a distinguished point* $x \in \mathrm{X}$. *Then the* skyscraper presheaf at x with value S *is defined by the rule*

$$\mathrm{S}^{x}(\mathrm{U}) := \begin{cases} \mathrm{S} & \textit{if} \quad x \in \mathrm{U}; \\ \star & \textit{otherwise.} \end{cases}$$

5.14. **Definition.** Suppose X a space, and suppose $\mathscr{F}$ a presheaf on X. Then for any point $x \in \mathrm{X}$, consider the set

$$\coprod_{x \in \mathrm{U} \in \mathrm{Op}(\mathrm{X})} \mathscr{F}(\mathrm{U}) = \{(\mathrm{U}, s) \mid x \in \mathrm{U} \in \mathrm{Op}(\mathrm{X}),\ s \in \mathscr{F}(\mathrm{U})\}.$$

On this set we may impose an equivalence relation $\sim$ in the following manner. For any two elements (U, s) and (V, t), we say that $(U, s) \sim (V, t)$ if and only if there exists an open neighborhood $W \subset U \cap V$ of x such that $\rho_{W \subset U}(s) = \rho_{W \subset V}(t)$. Now define the *stalk* of $\mathscr{F}$ at x to be the set

$$\mathscr{F}_x := \left(\coprod_{x \in U \in \mathrm{Op}(X)} \mathscr{F}(U) \right) \Big/ \sim .$$

The equivalence class of a section s under this equivalence relation is called the *germ* of s, and is denoted s_x.

5.15. **Lemma.** *Suppose* X *a space, and suppose* $x \in X$ *a point. For any morphism* $\phi \colon \mathscr{F} \longrightarrow \mathscr{G}$ *of presheaves on* X*, the set map*

$$\coprod_{x \in U \in \mathrm{Op}(X)} \mathscr{F}(U) \longrightarrow \coprod_{x \in U \in \mathrm{Op}(X)} \mathscr{G}(U)$$

descends to a set map on the stalks $\phi_x \colon \mathscr{F}_x \longrightarrow \mathscr{G}_x$.

5.16. **Proposition.** *Suppose* X *a space, and suppose* $x \in X$ *a point. Then for any presheaf* $\mathscr{F}$ *on* X *and any set* S*, there is a natural isomorphism*

$$\mathrm{Map}(\mathscr{F}_x, S) \cong \mathrm{Mor}(\mathscr{F}, S^x).$$

5.17. **Notation.** Suppose U a space, and suppose $\{U_\alpha\}_{\alpha \in \Lambda}$ an open cover of U. For any $\eta, \theta \in \Lambda$, write

$$U_{\eta\theta} := U_\eta \cap U_\theta.$$

5.18. **Definition.** A presheaf $\mathscr{F}$ on a space X is said to be a *sheaf* if, for any open set $U \in \mathrm{Op}(X)$ and any open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of U, the map

$$\prod_{\alpha \in \Lambda} \rho_{U_\alpha \subset U} \colon \mathscr{F}(U) \longrightarrow \prod_{\alpha \in \Lambda} \mathscr{F}(U_\alpha)$$

is an injection that identifies $\mathscr{F}(U)$ with the set of tuples

$$(s_\alpha)_{\alpha \in \Lambda} \in \prod_{\alpha \in \Lambda} \mathscr{F}(U_\alpha)$$

such that for any $\eta, \theta \in \Lambda$, one has

$$\rho_{U_{\eta\theta} \subset U_\eta}(s_\eta) = \rho_{U_{\eta\theta} \subset U_\theta}(s_\theta).$$

5.19. **Lemma.** *Suppose* $\mathscr{F}$ *a sheaf on a space* X*. Then* $\mathscr{F}(\varnothing) = \{*\}$.

5.20. **Example.** *Any sheaf on the one-point space $\{*\}$ is uniquely determined (up to isomorphism) by its set of global sections, so we will make no distinction between sets and sheaves on $\{*\}$.*

5.21. **Example.** *For any two spaces* X *and* Y*, the presheaf* $\mathscr{O}_X^Y$ *is a sheaf, called the* sheaf of local continuous functions on X with values in Y.

5.22. **Example.** *For any continuous map* $p\colon Y \longrightarrow X$*, the presheaf of local sections* $\Gamma(Y/X)$ *is a sheaf, called the* sheaf of local sections of p.

5.23. **Example.** *For any space* X*, any point* $x \in X$*, and any set* S*, the skyscraper presheaf* S^x *is a sheaf, called the* skyscraper sheaf.

5.24. **Example.** *For any space* X *and any open set* $U \in \mathrm{Op}(X)$*, the presheaf* $\mathscr{H}_U$ *is a sheaf, called the* sheaf represented by U.

5.25. **Example.** *For any space* X *and any set* $S \neq \{*\}$*, the constant presheaf on* X *at* S *is not a sheaf.*

5.26. **Theorem.** *Suppose* X *a space, and suppose* $\mathscr{F}$ *a sheaf on* X*. For any open set* $U \in \mathrm{Op}(X)$*, the map*

$$\mathscr{F}(U) \longrightarrow \prod_{x \in U} \mathscr{F}_x$$

that carries a section s *to the equivalence class of the pair* (U, s) *is injective.*

5.26.1. **Corollary.** *Suppose* $\mathscr{F}$ *and* $\mathscr{G}$ *sheaves on a space* X*. Then if*

$$\phi,\ \psi\colon \mathscr{F} \longrightarrow \mathscr{G}$$

are two morphisms such that for every point $x \in X$*, the induced maps*

$$\phi_x,\ \psi_x\colon \mathscr{F}_x \longrightarrow \mathscr{G}_x$$

on stalks coincide (so that $\phi_x = \psi_x$*), then* $\phi = \psi$*.*

5.27. **Theorem.** *Suppose* $\mathscr{F}$ *and* $\mathscr{G}$ *sheaves on a space* X*. Then a morphism*

$$\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$$

is a bijection or an injection if and only if, for every point $x \in X$*, the set map on stalks* $\phi_x\colon \mathscr{F}_x \longrightarrow \mathscr{G}_x$ *is so.*

5.28. **Warning.** If $\mathscr{F}$ and $\mathscr{G}$ are sheaves on a space X such that there are bijections $\mathscr{F}_x \cong \mathscr{G}_x$ for every point $x \in X$, it does *not* follow that $\mathscr{F}$ and $\mathscr{G}$ are isomorphic.

5.29. **Notation.** Suppose X a space and $\mathscr{F}$ a presheaf on X. Consider the set

$$\text{Ét}(\mathscr{F}) := \coprod_{x \in X} \mathscr{F}_x;$$

there is an obvious map $p_{\mathscr{F}} \colon \text{Ét}(\mathscr{F}) \longrightarrow X$ whose fibers are precisely the stalks of $\mathscr{F}$. For any open set U and any section $s \in \mathscr{F}(U)$, there is a corresponding map

$$\sigma_s \colon U \longrightarrow \text{Ét}(\mathscr{F}),$$

given by the assignment $x \longmapsto s_x$, such that $p \circ s = \mathrm{id}$.

5.30. **Definition.** Suppose X a space and $\mathscr{F}$ a presheaf on X. The *espace étalé* of $\mathscr{F}$ is the set $\text{Ét}(\mathscr{F})$ equipped with the finest topology such that for any section $s \in \mathscr{F}(U)$, the corresponding map

$$\sigma_s \colon U \longrightarrow \text{Ét}(\mathscr{F})$$

is continuous. That is, we declare a subset $V \subset \text{Ét}(\mathscr{F})$ to be open if and only if, for any open set $U \in \mathrm{Op}(X)$ and any section $s \in \mathscr{F}(U)$, the inverse image $\sigma_s^{-1}(V)$ is open in U.

5.31. **Definition.** A continuous map $p \colon Y \longrightarrow X$ is said to be a *local homeomorphism* if every point $y \in Y$ is contained in a neighborhood V such that p is open and injective.

5.32. **Proposition.** *For any space* X *and any presheaf* $\mathscr{F}$ *on* X*, the natural morphism* $p_{\mathscr{F}} \colon \text{Ét}(\mathscr{F}) \longrightarrow X$ *is a local homeomorphism.*

5.33. **Proposition.** *Suppose* S *a set, and suppose* $\mathscr{P}_S$ *is the constant presheaf at* S *on a space* X*. Then the éspace étalé of* $\mathscr{P}_S$ *is the projection* $\mathrm{pr}_1 \colon X \times S^{\delta} \longrightarrow X$.

5.34. **Lemma.** *Suppose* $p \colon Y \longrightarrow X$ *a local homeomorphism. Then the éspace étalé* $Z := \text{Ét}(\Gamma(Y/X))$ *of the sheaf of local sections* $\Gamma(Y/X)$ *is canonically homeomorphic* over X *to* Y*. That is, there is a unique homeomorphism* $Y \longrightarrow Z$ *such that the diagram*

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

commutes.

5.35. **Definition.** Suppose X a space and $\mathscr{F}$ a presheaf on X. The *sheafification* of $\mathscr{F}$ is the sheaf

$$a\mathscr{F} := \Gamma(\text{Ét}(\mathscr{F})/X)$$

of local sections of the projection map $p\colon \text{Ét}(\mathscr{F}) \longrightarrow X$. The morphism of presheaves

$$\eta_{\mathscr{F}}\colon \mathscr{F} \longrightarrow a\mathscr{F}$$

that assigns to any section $s \in \mathscr{F}(U)$ the section $x \longmapsto s_x$ over U is called the *unit morphism*.

5.36. **Proposition.** *For any presheaf $\mathscr{F}$ on a space* X*, the natural morphism*

$$\eta_{\mathscr{F}}\colon \mathscr{F} \longrightarrow a\mathscr{F}$$

induces a bijection $\eta_{\mathscr{F},x}\colon \mathscr{F}_x \longrightarrow (a\mathscr{F})_x$ on stalks for every $x \in X$.

5.36.1. **Corollary.** *For any sheaf $\mathscr{F}$ on a space* X*, the unit morphism $\eta_{\mathscr{F}}\colon \mathscr{F} \longrightarrow a\mathscr{F}$ is an isomorphism.*

5.37. **Example.** *The* constant sheaf $\mathscr{F}_S$ *at a set* S *on a space* X *is the sheafification of the constant presheaf $\mathscr{P}_S$ at* S*. It is isomorphic to the sheaf of local sections $\Gamma(X \times S^{\delta}/X)$. Consequently, the constant sheaf is not really constant: it takes many different values on an open set $U \subset X$.*

5.38. **Proposition.** *Suppose* X *a numerically generated space. Then there exists a global section $u \in \mathscr{F}_{\pi_0 X}(X)$ such that for any set* S *and any global section $\sigma \in \mathscr{F}_S(X)$, there exists a unique set map $\pi_0 \longrightarrow S$ such that the induced morphism of sheaves $\widetilde{\sigma}\colon \mathscr{F}_{\pi_0 X} \longrightarrow \mathscr{F}_S$ has the property that $\widetilde{\sigma}(u) = \sigma$.*

5.39. **Theorem.** *Suppose* X *a space, suppose $\mathscr{F}$ a presheaf on* X*, and suppose $\mathscr{G}$ a sheaf on* X*. Then for any morphism $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$, there exists a unique morphism $a\phi\colon a\mathscr{F} \longrightarrow \mathscr{G}$ such that the diagram*

$$\begin{array}{ccc} & \mathscr{F} & \\ {}^{\eta_{\mathscr{F}}}\swarrow & & \searrow^{\phi} \\ a\mathscr{F} & \xrightarrow[a\phi]{} & \mathscr{G} \end{array}$$

commutes.

5.39.1. **Corollary.** *Suppose* X *a space. For any morphism $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ of presheaves, there exists a unique morphism*

$$a\phi\colon a\mathscr{F} \longrightarrow a\mathscr{G}$$

such that following diagram commutes:

$$\begin{array}{ccc} \mathscr{F} & \xrightarrow{\phi} & \mathscr{G} \\ {\scriptstyle \eta_{\mathscr{F}}}\downarrow & & \downarrow{\scriptstyle \eta_{\mathscr{G}}} \\ a\mathscr{F} & \xrightarrow[a\phi]{} & a\mathscr{G}. \end{array}$$

5.40. **Definition.** Suppose $g\colon \mathrm{X} \longrightarrow \mathrm{Y}$ a map.

(5.40.1) For any sheaf $\mathscr{F}$ on X, define the *direct image* $g_\star\mathscr{F}$ of $\mathscr{F}$ as the sheaf that assigns to any open set $\mathrm{V} \in \mathrm{Op}(\mathrm{Y})$ the set

$$g_\star\mathscr{F}(\mathrm{V}) := \mathscr{F}(g^{-1}\mathrm{V}).$$

(5.40.2) For any sheaf $\mathscr{G}$ on Y, we define the *inverse image* $g^\star\mathscr{G}$ as the sheaf of local sections of the pullback

$$\mathrm{X} \times_{\mathrm{Y}} \mathrm{\acute{E}t}(\mathscr{G}) \longrightarrow \mathrm{X}$$

of the map $p_{\mathscr{G}} : \mathrm{\acute{E}t}(\mathscr{G}) \longrightarrow \mathrm{Y}$

5.41. **Example.** *Suppose* $\mathrm{A} \subset \mathrm{X}$ *a subspace of a space* X. *Then for any sheaf* $\mathscr{F}$ *on* X, *if i denotes the inclusion map, the sheaf* $i^\star\mathscr{F}$ *on* A *is denoted* $\mathscr{F}|_{\mathrm{A}}$ *and is called the* restriction *of* $\mathscr{F}$ *to* A. *If, in particular,* A *is an open set, then the restriction* $\mathscr{F}|_{\mathrm{A}}$ *assigns to any open set* $\mathrm{U} \subset \mathrm{A}$ *the set* $\mathscr{F}(\mathrm{U})$.

5.42. **Example.** *Suppose* X *a space. We have a unique map* $!\colon \mathrm{X} \longrightarrow \{*\}$. *For any set* S, *there is a natural isomorphism*

$$!^\star\mathrm{S} \cong \mathscr{F}_{\mathrm{S}}$$

between the inverse image along ! *and the constant sheaf. For any sheaf* $\mathscr{F}$ *on* X, *there is a natural isomorphism*

$$!_\star\mathscr{F} \cong \mathscr{F}(\mathrm{X})$$

between the direct image along ! *and the set of global sections.*

On the other hand, suppose $x \in \mathrm{X}$ *a point, and write* $x\colon \{*\} \longrightarrow \mathrm{X}$ *for the corresponding inclusion. For any set* S, *there is a natural isomorphism*

$$x_\star\mathrm{S} \cong \mathrm{S}^x$$

between the direct image along x and the skyscraper sheaf. For any sheaf $\mathscr{F}$ *on* X, *there is a natural isomorphism*

$$x^\star\mathscr{F} \cong \mathscr{F}_x$$

between the inverse image along x and the stalk of $\mathscr{F}$ *at x.*

5.43. **Theorem.** *Suppose* $g\colon \mathrm{X} \longrightarrow \mathrm{Y}$ *a map,* $\mathscr{F}$ *a sheaf on* X *and* $\mathscr{G}$ *a sheaf on* Y. *Then there exists a natural bijection*

$$\mathrm{Mor}_{\mathrm{X}}(g^\star \mathscr{G}, \mathscr{F}) \cong \mathrm{Mor}_{\mathrm{Y}}(\mathscr{G}, g_\star \mathscr{F}).$$

5.44. **Definition.** Suppose X a space. A sheaf $\mathscr{F}$ on X will be said to be *locally constant* if every point $x \in \mathrm{X}$ is contained in an open neighborhood U such that the sheaf $\mathscr{F}|_{\mathrm{U}}$ is constant.

5.45. **Notation.** For any space X, denote by **LC**(X) the category whose objects are locally constant sheaves on X and whose morphisms are morphisms of sheaves.

5.46. **Example.** *For any natural number n, consider the map* $p_n\colon \mathbf{C}^\times \longrightarrow \mathbf{C}^\times$ *given by* $\xi \longmapsto \xi^n$. *Then the sheaf of local sections* $\Gamma(p_n)$ *is locally constant, but it is not constant.*

5.47. **Proposition.** *Suppose* X *a connected numerically generated space. Then a locally constant sheaf* $\mathscr{F}$ *on* X *is a constant sheaf if and only if for any point* $x \in \mathrm{X}$, *the set map* $\mathscr{F}(\mathrm{X}) \longrightarrow \mathscr{F}_x$ *that carries a global section s to its equivalence class in* $\mathscr{F}_x$ *is a bijection.*

5.48. **Proposition.** *The only locally constant sheaves on* I *are constant.*

5.49. **Definition.** Suppose X a space. Write **Set** for the category whose objects are sets and whose morphisms are set maps. For any point $x \in \mathrm{X}$, the *fiber functor* for x is the functor $\omega_x := x^\star\colon \mathbf{LC}(\mathrm{X}) \longrightarrow \mathbf{Set}$.

5.50. **Notation.** Suppose X a numerically generated space, suppose $x, y \in \mathrm{X}$, and suppose $\gamma\colon \mathrm{I} \longrightarrow \mathrm{X}$ a path such that $\gamma(0) = x$ and $\gamma(1) = y$. If $\mathscr{F}$ is a locally constant sheaf on X, then we obtain a bijection $\omega_\gamma(\mathscr{F})$:

$$\omega_x(\mathscr{F}) \cong (\gamma^\star\mathscr{F})_0 \xleftarrow{\cong} (\gamma^\star\mathscr{F})(\mathrm{I}) \xrightarrow{\cong} (\gamma^\star\mathscr{F})_1 \cong \omega_y(\mathscr{F}).$$

5.51. **Proposition.** *Suppose* X *a numerically generated space, suppose* $x, y \in \mathrm{X}$, *and suppose* $\gamma\colon \mathrm{I} \longrightarrow \mathrm{X}$ *a path such that* $\gamma(0) = x$ *and* $\gamma(1) = y$. *If* $\phi\colon \mathscr{F} \longrightarrow \mathscr{G}$ *is a morphism of locally constant sheaves on* X, *one has*

$$\phi_y \circ \omega_\gamma(\mathscr{F}) = \omega_\gamma(\mathscr{G}) \circ \phi_x.$$

5.51.1. **Corollary.** *Suppose* X *a numerically generated space, suppose* $x, y \in \mathrm{X}$, *and suppose* $\gamma\colon \mathrm{I} \longrightarrow \mathrm{X}$ *a path such that* $\gamma(0) = x$ *and* $\gamma(1) = y$. *Then* ω_γ *is a natural isomorphism* $\omega_x \xrightarrow{\sim} \omega_y$.

5.52. **Proposition.** *Suppose* X *a numerically generated space, and suppose* $x, y \in$ X. *If* $\eta, \theta \in \mathrm{P}_{x,y}(\mathrm{X})$ *lie in the same connected component, then one has*

$$\omega_\eta = \omega_\theta.$$

5.53. **Definition.** Suppose X a space. Write **Fib**(X) for the following groupoid. The objects are points $x \in \mathrm{X}$, and for any two points $x, y \in \mathrm{X}$, the set

$$\mathrm{Isom}_{\mathbf{Fib}(\mathrm{X})}(x, y)$$

is the set of natural isomorphisms $\omega_x \xrightarrow{\sim} \omega_y$.

5.54. **Definition.** Suppose X a numerically generated space. Then we say that X is *locally contractible* if, for any point $x \in \mathrm{X}$ and any open neighborhood U of x, there exists a neighborhood $x \in \mathrm{V} \subset \mathrm{U}$ such that the inclusion $\{x\} \hookrightarrow \mathrm{V}$ is a homotopy equivalence.

5.55. **Theorem.** *Suppose* X *a locally contractible numerically generated space. Then the assignment* $\gamma \longmapsto \omega_\gamma$ *defines an equivalence of groupoids*

$$\Pi_1(\mathrm{X}) \xrightarrow{\sim} \mathbf{Fib}(\mathrm{X}).$$

5.55.1. **Corollary.** *For any locally contractible numerically generated space* X *and any point* $x \in \mathrm{X}$*, the assignment* $\gamma \longmapsto \omega_\gamma$ *defines an isomorphism*

$$\pi_1(\mathrm{X}, x) \xrightarrow{\sim} \mathrm{Aut}(\omega_x).$$

6. Simplicial sets and higher groupoids

6.1. **Definition.** Consider the following category Δ. The objects are nonempty totally ordered finite sets, and a morphism $\mathrm{K} \longrightarrow \mathrm{J}$ in Δ is a nondecreasing map $\mathrm{K} \longrightarrow \mathrm{J}$.

For any natural number n, denote by $[n]$ the totally ordered finite set

$$\{0, \ldots, n\}$$

(whose order is the usual one). We regard $[n]$ as an object of Δ.

6.2. **Lemma.** *For every object* J *of* Δ*, there exists a unique integer* n_J *and a unique isomorphism* $\mathrm{J} \xrightarrow{\sim} [n_\mathrm{J}]$. *For any two objects* J *and* K *of* Δ*, the set* $\mathrm{Isom}_\Delta(\mathrm{K}, \mathrm{J})$ *of isomorphisms* $\mathrm{K} \xrightarrow{\sim} \mathrm{J}$ *is given by*

$$\mathrm{Isom}_\Delta(\mathrm{K}, \mathrm{J}) \cong \begin{cases} \{*\} & \textit{if } n_\mathrm{K} = n_\mathrm{J} \\ \varnothing & \textit{if } n_\mathrm{K} \neq n_\mathrm{J}. \end{cases}$$

6.3. **Lemma.** *Every morphism* $g\colon K \longrightarrow J$ *of* Δ *can be factored in a unique fashion as* $g = g_+ \circ g_-$, *where* g_+ *is an injective nondecreasing map, and* g_- *is a surjective nondecreasing map.*

6.4. **Lemma.** *Suppose n a natural number. For any integer* $0 \leq i \leq n$, *there is a unique nondecreasing injection*

$$\delta_i\colon [n-1] \longrightarrow [n]$$

such that i is not contained in the image of δ_i. *Similarly, there is a unique nondecreasing surjection*

$$\sigma_i\colon [n+1] \longrightarrow [n]$$

such that $\sigma_i(i) = \sigma_i(i+1)$.

6.5. **Definition.** A *simplicial set* is a functor $X\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$. The set $X([n])$ will usually be denoted X_n. Its elements will be called *n-simplices*. We sometimes call 0-simplices *vertices* and 1-simplices *edges*.

A *morphism* $g\colon X \longrightarrow Y$ of simplicial sets is a natural transformation. That is, it is a tuple $(g_J)_{J\in\Delta}$ of set maps $g_J\colon X(J) \longrightarrow Y(J)$ such that for any morphism $\phi\colon K \longrightarrow J$ of Δ, the diagram

$$\begin{array}{ccc} X(J) & \xrightarrow{g_J} & Y(J) \\ {\scriptstyle X(\phi)}\downarrow & & \downarrow{\scriptstyle Y(\phi)} \\ X(K) & \xrightarrow[g_K]{} & Y(K) \end{array}$$

commutes. We write $\mathrm{Mor}(X, Y)$ for the set of morphisms $X \longrightarrow Y$.

6.6. **Lemma.** *A simplicial set* X *is uniquely identified by the following data:*

(6.6.A) for any natural number n, a set X_n*;*
(6.6.B) for any natural number n and any integer $0 \leq i \leq n$, *a map* $d_i := X(\delta_i)\colon X_n \longrightarrow X_{n-1}$*;*
(6.6.C) for any natural number n and any integer $0 \leq i \leq n$, *a map* $s_i := X(\sigma_i)\colon X_n \longrightarrow X_{n+1}$*;*

subject to the following axioms.

(6.6.1) If $i < j$, *then* $d_i d_j = d_{j-1} d_i$.
(6.6.2) If $i > j$, *then* $s_i s_j = s_j s_{i-1}$.
(6.6.3) Lastly,

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j; \\ \mathrm{id} & \text{if } i = j \text{ or } i = j+1; \\ s_j d_{i-1} \text{if } i > j+1. \end{cases}$$

6.7. **Example.** *For any set* S*, the* discrete simplicial set S^{δ} at S *is constant functor*

$$J \longmapsto S.$$

6.8. **Example.** *For any object* J *of* Δ*, the simplicial set* Δ^J *is given by the assignment*

$$K \longmapsto \mathrm{Mor}_{\Delta}(K, J).$$

For any simplicial set X*, there is a natural bijection*

$$\mathrm{Mor}(\Delta^J, X) \cong X(J).$$

For any natural number n*, we write* Δ^n *for* $\Delta^{[n]}$*, and we call it the* standard n-simplex.

6.9. **Example.** *For any category* C*, the* nerve NC *is defined in the following manner. Any object* J *of* Δ *can be regarded as a category whose objects are the elements of* I *and whose morphisms are given by*

$$\mathrm{Mor}_J(i, j) \cong \begin{cases} \{*\} & \text{if } i \leq j \\ \varnothing & \text{if } i > j. \end{cases}$$

Now NC *is given by the assignment*

$$J \longmapsto \mathrm{Fun}(J, C),$$

where $\mathrm{Fun}(J, C)$ *denotes the set of functors* $J \longrightarrow C$.

6.10. **Lemma.** *For any object* J *of* Δ*, there is a natural isomorphism* $NJ \cong \Delta^J$.

6.11. **Proposition.** *For any categories* C *and* D*, the natural map*

$$\mathrm{Fun}(C, D) \longrightarrow \mathrm{Mor}(NC, ND)$$

is a bijection.

6.12. **Example.** *For any two simplicial sets* X *and* Y*, the* product $X \times Y$ *is the functor given by the assignment*

$$J \longmapsto X(J) \times Y(J).$$

More generally, for any morphisms $X \longrightarrow Z$ *and* $Y \longrightarrow Z$ *of simplicial sets, the* fiber product $X \times_Z Y$ *is the functor given by the assignment*

$$J \longmapsto X(J) \times_{Z(J)} Y(J).$$

6.13. **Example.** *If* X *and* Y *are two simplicial sets, then the* coproduct $X \sqcup Y$ *is the functor given by the assignment*

$$J \longmapsto X(J) \sqcup Y(J).$$

6.14. **Definition.** Suppose X a simplicial set. Suppose n a natural number and $\tau \in X_n$. For an integer $0 \leq i \leq n$, the *i-th face* of τ is the $(n-1)$-simplex $d_i(\tau)$, and the *i-th degeneracy* of τ is the $(n+1)$-simplex $s_i(\tau)$.

An $(n+1)$-simplex is *degenerate* if it lies in the essential image of $X(\sigma_i)$; we'll say that it is *nondegenerate* if it is not degenerate.

6.15. **Lemma.** *Suppose* X *a simplicial set, and suppose that for every natural number n, one has a subset* $Y_n \subset X_n$. *If* $\tau \in Y_n$ *implies that for any integer* $0 \leq i \leq n$, *one has* $d_i(\tau) \in Y_{n-1}$ *and* $s_i(\tau) \in Y_{n+1}$, *then the assignment* $I \longmapsto Y_{n_I}$ *defines a simplicial set, and the inclusions* $Y_n \hookrightarrow X_n$ *define a morphism of simplicial sets.*

6.16. **Definition.** A simplicial set Y constructed as above will be called a *simplicial subset* of X, and we will write $Y \subset X$.

6.17. **Example.** *For any morphisms* $X \longrightarrow Z$ *and* $Y \longrightarrow Z$ *of simplicial sets, the fiber product* $X \times_Z Y$ *is naturally a simplicial subset of* $X \times Y$.

6.18. **Example.** *For any natural number n and any integer* $0 \leq i \leq n$, *the inclusion*

$$\{0, \ldots, i-1, i+1, \ldots, n\} \hookrightarrow [n]$$

defines a simplicial subset

$$\Delta^{\{0,\ldots,i-1,i+1,\ldots,n\}} \subset \Delta^n,$$

*which we call the i-*th face of Δ^n.

6.19. **Example.** *For any natural number n, denote by* $\partial\Delta^n \subset \Delta^n$ *the smallest simplicial subset that contains all the faces of* Δ^n. *That is, the set of m-simplices of* $\partial\Delta^n$ *is given by*

$$(\partial\Delta^n)_m := \bigcup_{0 \leq i \leq n} \Delta_m^{\{0,\ldots,i-1,i+1,\ldots,n\}}.$$

6.20. **Example.** *For any natural number n and any integer* $0 \leq k \leq n$, *denote by* $\Lambda_k^n \subset \partial\Delta^n$ *the smallest simplicial subset that contains all the faces of* Δ^n *except for the k-th. That is, the set of m-simplices of* Λ_k^n *is given by*

$$(\Lambda_k^n)_m := \bigcup_{0 \leq i \leq n,\ i \neq k} \Delta_m^{\{0,\ldots,i-1,i+1,\ldots,n\}}.$$

6.21. **Example.** *For any simplicial set* X *and any integer* $n \geq 0$, *let* $\mathrm{sk}_n X \subset X$ *be the smallest simplicial subset of* X *that contains all the n-simplices of* X. *That is, the* *set of m-simplices of* $\mathrm{sk}_n \mathrm{X}$ *is given by*

$$(\mathrm{sk}_n \mathrm{X})_m := \begin{cases} \mathrm{X}_m & \textit{if } m \leq n; \\ \bigcup_{i_1,\dots,i_{m-n}} s_{i_{m-n}} \cdots s_{i_1}(\mathrm{X}_m) & \textit{if } m > n. \end{cases}$$

6.22. **Definition.** Suppose n a natural number, and write $\Delta_{\leq n} \subset \Delta$ for the full subcategory spanned by those objects J of Δ such that $n_\mathrm{J} \leq n$. For any simplicial set X, write $\mathrm{X}_{\leq n}$ for the restriction of X to $\Delta^{\mathrm{op}}_{\leq n}$.

6.23. **Lemma.** *For any natural number n, one has*

$$\mathrm{sk}_{n-1} \Delta^n \cong \partial\Delta^n,$$

and for any integer $0 \leq k \leq n+1$,

$$\mathrm{sk}_{n-1} \Lambda^{n+1}_k \cong \mathrm{sk}_{n-1} \Delta^{n+1}.$$

6.24. **Definition.** For any natural number n and any simplicial set X, define a simplicial set $\mathrm{cosk}_n \mathrm{X}$ as the functor given by the assignment

$$\mathrm{J} \longmapsto \mathrm{Mor}(\mathrm{sk}_n \Delta^\mathrm{J}, \mathrm{X}).$$

The inclusions $\mathrm{sk}_n \Delta^\mathrm{J} \hookrightarrow \Delta^\mathrm{J}$ induce a morphism $\mathrm{X} \longrightarrow \mathrm{cosk}_n \mathrm{X}$. We say that X is *n-coskeletal* if this morphism is an isomorphism.

6.25. **Proposition.** *For any natural number n and any two simplicial sets* X *and* Y*, there are natural bijections*

$$\mathrm{Mor}(\mathrm{sk}_n \mathrm{X}, \mathrm{Y}) \cong \mathrm{Nat}(\mathrm{X}_{\leq n}, \mathrm{Y}_{\leq n}) \cong \mathrm{Mor}(\mathrm{X}, \mathrm{cosk}_n \mathrm{Y}),$$

where $\mathrm{Nat}(\mathrm{X}_{\leq n}, \mathrm{Y}_{\leq n})$ *is the set of natural transformations* $\mathrm{X}_{\leq n} \longrightarrow \mathrm{Y}_{\leq n}$.

6.26. **Proposition.** *The nerve of any category is* 2*-coskeletal.*

6.27. **Definition.** A simplicial set X is a *Kan complex* or an ∞*-groupoid* if for any natural number $n \geq 1$ and any integer $0 \leq k \leq n$, the inclusion morphism $\Lambda^n_k \hookrightarrow \Delta^n$ induces a surjection

$$\mathrm{Mor}(\Delta^n, \mathrm{X}) \longrightarrow \mathrm{Mor}(\Lambda^n_k, \mathrm{X}).$$

For a natural number m, we say that an ∞-groupoid X is a *m-groupoid* if, in addition, for any natural number $n \geq m+1$ and any integer $0 \leq k \leq n$, the inclusion morphism $\Lambda^n_k \hookrightarrow \Delta^n$ induces a bijection

$$\mathrm{X}_n \cong \mathrm{Mor}(\Delta^n, \mathrm{X}) \longrightarrow \mathrm{Mor}(\Lambda^n_k, \mathrm{X}).$$

6.28. **Example.** *The standard simplex* Δ^n *is a Kan complex if and only if* $n = 0$.

6.29. **Example.** *A* 0*-groupoid is precisely a discrete simplicial set.*

6.30. **Proposition.** *The nerve of a category* C *is a Kan complex if and only* C *is a groupoid, in which case* NC *is a* 1*-groupoid.*

6.31. **Proposition.** *An m-groupoid is* $(m+1)$*-coskeletal, and an m-coskeletal Kan complex is a* $(m+1)$*-groupoid.*

6.32. **Proposition.** *Any* 1*-groupoid is the nerve of a groupoid.*

6.33. **Proposition.** *If* X *and* Y *are m-groupoids (*$0 \leq m \leq \infty$*), then the product* $X \times Z$ *is an m-groupoid as well.*

6.34. **Proposition.** *If* X *and* Y *are m-groupoids (*$0 \leq m \leq \infty$*), then the coproduct* $X \sqcup Y$ *is an m-groupoid as well.*

6.35. **Proposition.** *Suppose* $X\colon \Delta^{op} \longrightarrow \mathbf{Grp}$ *a* simplicial group*, i.e., a simplicial set in which each* X_n *is equipped with a group structure and the maps* $d_i\colon X_n \longrightarrow X_{n-1}$ *and* $s_i\colon X_n \longrightarrow X_{n+1}$ *are all group homomorphisms. Then* X *is a Kan complex.*

6.36. **Definition.** Suppose X and Y two simplicial sets. Define a simplicial set $\mathrm{Map}(X, Y)$ as the functor given by the assignment

$$J \longmapsto \mathrm{Mor}(X \times \Delta^J, Y).$$

6.37. **Lemma.** *For any simplicial sets* X*,* Y*, and* Z*, there is a natural bijection*

$$\mathrm{Mor}(X \times Y, Z) \cong \mathrm{Mor}(X, \mathrm{Map}(Y, Z)).$$

6.38. **Proposition.** *Suppose* C *and* D *two categories. Then there is a natural isomorphism*

$$N\mathbf{Fun}(C, D) \cong \mathrm{Map}(NC, ND),$$

where $\mathbf{Fun}(C, D)$ *denotes the category whose objects are functors* $C \longrightarrow D$ *and whose morphisms are natural transformations.*

6.39. **Theorem.** *Suppose* X *a simplicial set, and suppose* Y *an m-groupoid (*$0 \leq m \leq \infty$*). Then* $\mathrm{Map}(X, Y)$ *is an m-groupoid as well.*

6.39.1. **Corollary.** *For any simplicial set* X *and for any groupoid* Γ*, the simplicial set* $\mathrm{Map}(X, N\Gamma)$ *is the nerve of a groupoid.*

6.39.2. **Corollary.** *For any simplicial set* X *and for any set* S*, the simplicial set* $\mathrm{Map}(X, S^\delta)$ *is discrete.*

7. The Postnikov tower

7.1. **Definition.** Suppose X a simplicial set. Consider the equivalence relation $\sim$ on X_0 generated by declaring two vertices $x, y \in X_0$ to be equivalent if there exists a 1-simplex $\tau \in X_1$ such that $d_0(\tau) = x$ and $d_1(\tau) = y$. Denote by $\pi_0 X := X/\sim$ the set of equivalence classes under this equivalence relation, and write $p_{X,0} \colon X_0 \longrightarrow \pi_0 X$ the projection of the vertices of X onto their equivalence classes.

7.2. **Example.** *For any set* S*, one has* $\pi_0(S^\delta) = S$*.*

7.3. **Lemma.** *If* X *is a Kan complex, then two vertices are equivalent in the sense above if and only if there exists a* 1*-simplex* $\tau \in X_1$ *such that* $d_0(\tau) = x$ *and* $d_1(\tau) = y$*.*

7.4. **Lemma.** *Suppose* X *a simplicial set. For any natural number* $n \geq 1$*, any* n*-simplex* $\tau \in X_n$*, and any two morphisms* $\phi, \psi \colon [0] \longrightarrow [n]$ *of* Δ*, we have*

$$X(\phi)(\tau) \sim X(\psi)(\tau).$$

Consequently, there exists a unique morphism $p_X \colon X \longrightarrow (\pi_0 X)^\delta$ *that on vertices is the map* $p_{X,0}$ *above.*

7.5. **Theorem.** *Suppose* X *a simplicial set. Then the morphism* p_X *has the following universal property: for any set* S *and any morphism* $g \colon X \longrightarrow S^\delta$*, there exists a unique set map* $\pi_0 g \colon \pi_0 X \longrightarrow S$ *such that the following diagram commutes:*

$$\begin{array}{ccc} & X & \\ {}^{p_X}\swarrow & & \searrow^{g} \\ (\pi_0 X)^\delta & \xrightarrow[\pi_0 g]{} & S^\delta. \end{array}$$

7.5.1. **Corollary.** *For any morphism* $g \colon X \longrightarrow Y$ *between simplicial sets, there exists a unique set map*

$$\pi_0 g \colon \pi_0 X \longrightarrow \pi_0 Y$$

such that following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ (\pi_0 X)^\delta & \xrightarrow[(\pi_0 g)^\delta]{} & (\pi_0 Y)^\delta. \end{array}$$

7.5.2. **Corollary.** *The assignment* $X \longmapsto \pi_0 X$ *defines a functor* s**Set** $\longrightarrow$ **Set** *that is left adjoint to the functor given by the assignment* $S \longmapsto S^\delta$.

7.6. **Proposition.** *For any simplicial sets* X *and* Y, *the two maps* $X \times Y \longrightarrow X$ *and* $X \times Y \longrightarrow Y$ *together induce a bijection*

$$\pi_0(X \times Y) \xrightarrow{\sim} \pi_0 X \times \pi_0 Y.$$

7.7. **Proposition.** *For any family* $\{X_i\}$ *of numerically generated spaces, the inclusions* $X_i \hookrightarrow \coprod_i X_i$ *together induce a bijection*

$$\coprod_i \pi_0(X_i) \cong \pi_0\left(\coprod_i X_i\right).$$

7.8. **Definition.** Suppose X a simplicial set and Y a Kan complex. We will say that two morphisms $p, q\colon X \longrightarrow Y$ are *homotopic* if the images of p and q in $\pi_0 \operatorname{Map}(X, Y)$ are equal. In this case we write $p \simeq q$.

7.9. **Lemma.** *Suppose* X *a simplicial set and* Y *a Kan complex. Two morphisms* $p, q\colon X \longrightarrow Y$ *of simplicial sets are homotopic just in case there exists a map*

$$h\colon X \times \Delta^1 \longrightarrow Y$$

such that one has

$$h|(X \times \Delta^{\{0\}}) = p \quad \textit{and} \quad h|(X \times \Delta^{\{1\}}) = q.$$

7.10. **Definition.** We say that a morphism $\phi\colon X \longrightarrow Y$ of simplicial sets is a *homotopy equivalence* if there exists a map $\psi\colon\ Y \longrightarrow X$ such that both $\psi \circ \phi \simeq \mathrm{id}_X$ and $\phi \circ \psi \simeq \mathrm{id}_Y$.

7.11. **Proposition.** *A homotopy equivalence* $X \longrightarrow Y$ *between simplicial sets induces a bijection*

$$\pi_0 X \xrightarrow{\sim} \pi_0 Y.$$

7.12. **Definition.** Suppose Y a Kan complex, and suppose $X' \subset X$ a simplicial subset. We say that p and q are *homotopic relative to* X' if there exists a morphism

$$h \colon X \times \Delta^1 \longrightarrow Y$$

such that

$$h|(X \times \Delta^{\{0\}}) = p \quad \text{and} \quad h|(X \times \Delta^{\{1\}}) = q,$$

and $h|(X' \times \Delta^1)$ factors through the projection $X' \times \Delta^1 \longrightarrow X'$.

7.13. **Definition.** Suppose X a Kan complex. Consider the equivalence relation $\sim_1$ on the set X_1 generated by declaring two 1-simplices $\tau, \upsilon \in X_1$ to be equivalent if the corresponding maps $\tau, \upsilon \colon \Delta^1 \longrightarrow X$ are homotopic relative to $\partial\Delta^1$.

Define a groupoid $\Pi_1 X$ as follows. The objects of $\Pi_1 X$ are vertices of X, and for any vertices $x, y \in X_0$, the set $\mathrm{Isom}_{\Pi_1 X}(x, y)$ is the set of equivalence classes of 1-simplices.

7.14. **Example.** *For any groupoid* Γ*, there is an isomorphism of groupoids*

$$\Gamma \cong \Pi_1(N\Gamma).$$

7.15. **Proposition.** *Suppose* X *a Kan complex. Then the following are equivalent for two* 1*-simplices* $\tau, \upsilon \in X_1$.

- *(7.15.1)* $\tau \sim_1 \upsilon$.
- *(7.15.2) There exists a 2-simplex* η *such that* $d_0(\eta) = \tau$ *and* $d_1(\eta) = \upsilon$*, and* $d_2(\eta)$ *is degenerate.*
- *(7.15.3) There exists a 2-simplex* η *such that* $d_1(\eta) = \tau$ *and* $d_0(\eta) = \upsilon$*, and* $d_2(\eta)$ *is degenerate.*
- *(7.15.4) There exists a 2-simplex* η *such that* $d_1(\eta) = \tau$ *and* $d_2(\eta) = \upsilon$*, and* $d_0(\eta)$ *is degenerate.*
- *(7.15.5) There exists a 2-simplex* η *such that* $d_2(\eta) = \tau$ *and* $d_1(\eta) = \upsilon$*, and* $d_0(\eta)$ *is degenerate.*

7.16. **Proposition.** *Suppose* X *a Kan complex. Then there exists a unique morphism* $p_X \colon X \longrightarrow N\Pi_1 X$ *of simplicial sets such that* $p_{X,0}$ *is the identity map from the set* X_0 *to the set of objects of* $\Pi_1 X$*, and* $p_{X,1}$ *is the projection from* $X_1 \longrightarrow X_1/\sim_1$.

7.17. **Theorem.** *Suppose* X *a simplicial set. Then the morphism* p_X *has the following universal property: for any groupoid* Γ *and any morphism* $g \colon X \longrightarrow N\Gamma$*, there exists a unique morphism of groupoids* $\Pi_1 g \colon \Pi_1 X \longrightarrow \Gamma$ *such that the following diagram* *commutes:*

$$\begin{array}{ccc} & X & \\ {}^{p_X}\swarrow & & \searrow{}^{g} \\ N\Pi_1 X & \xrightarrow[\Pi_1 g]{} & \Gamma. \end{array}$$

7.17.1. **Corollary.** *For any morphism* $g\colon X \longrightarrow Y$ *between simplicial sets, there exists a unique morphism of groupoids*

$$\Pi_1 g\colon \Pi_1 X \longrightarrow \Pi_1 Y$$

such that following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ N\Pi_1 X & \xrightarrow[N\Pi_1 g]{} & \Pi_1 Y. \end{array}$$

7.17.2. **Corollary.** *The assignment* $X \longmapsto \Pi_1 X$ *defines a functor* **Kan** $\longrightarrow$ **Gpd** *that is left adjoint to the functor given by the assignment* $\Gamma \longmapsto N\Gamma$.

7.18. **Proposition.** *For any simplicial sets* X *and* Y*, the two maps* $X \times Y \longrightarrow X$ *and* $X \times Y \longrightarrow Y$ *together induce an isomorphism*

$$\Pi_1(X \times Y) \xrightarrow{\sim} \Pi_1 X \times \Pi_1 Y.$$

7.19. **Proposition.** *For any family* $\{X_i\}$ *of numerically generated spaces, the inclusions* $X_i \hookrightarrow \coprod_i X_i$ *together induce an isomorphism*

$$\coprod_i \Pi_1(X_i) \cong \Pi_1\left(\coprod_i X_i\right).$$

7.20. **Definition.** Suppose X a Kan complex, and suppose m a natural number. Consider the morphism $\operatorname{cosk}_{m+1} X \longrightarrow \operatorname{cosk}_m X$, and consider the simplicial subset $X^{(m)} \subset \operatorname{cosk}_m X$ whose set of k-simplices is the image of the set map $(\operatorname{cosk}_{m+1} X)_k \longrightarrow (\operatorname{cosk}_m X)_k$.

Now let $\sim_m$ be the equivalence relation on the simplices of $X^{(m)}$ generated by declaring two k-simplices $\tau, \upsilon \in (X^{(m)})_k$ to be equivalent if the corresponding morphisms $\tau, \upsilon\colon \operatorname{sk}_m \Delta^k \longrightarrow X$ are homotopic relative to $\operatorname{sk}_{m-1} \Delta^k$.

Now let $\Pi_m X$ denote the simplicial set whose k simplices are given by the set of equivalence classes

$$(\Pi_m X)_k := (X^{(m)})/\sim_k .$$

There is a natural morphism $X \longrightarrow X^{(m)}$, and thus a morphism $p_X\colon X \longrightarrow \Pi_m X$.

7.21. **Proposition.** *For any Kan complex* X*, the simplicial set* $\Pi_m X$ *is an m-groupoid.*

7.22. **Example.** *For any m-groupoid* X*, there is an isomorphism*

$$X \cong \Pi_m X.$$

7.23. **Example.** *For any Kan complex* X*, one has*

$$\Pi_0 X \cong (\pi_0 X)^{\delta}.$$

7.24. **Example.** *For any Kan complex* X*, one has*

$$\Pi_1 X \cong N\Pi_1 X.$$

(Note the abuse of notation.)

7.25. **Theorem.** *Suppose* X *a simplicial set and m a natural number. Then the morphism* p_X *has the following universal property: for any m-groupoid* Y *and any morphism* $g\colon X \longrightarrow Y$*, there exists a unique morphism of groupoids* $\Pi_m g\colon \Pi_m X \longrightarrow Y$ *such that the following diagram commutes:*

$$\begin{array}{ccc} & X & \\ {\scriptstyle p_X}\swarrow & & \searrow{\scriptstyle g} \\ \Pi_m X & \xrightarrow[\Pi_m g]{} & Y. \end{array}$$

7.25.1. **Corollary.** *For any natural number m and any morphism* $g\colon X \longrightarrow Y$ *between simplicial sets, there exists a unique morphism of m-groupoids*

$$\Pi_m g\colon \Pi_m X \longrightarrow \Pi_m Y$$

such that following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle p_Y} \\ \Pi_m X & \xrightarrow[\Pi_m g]{} & \Pi_m Y. \end{array}$$

7.25.2. **Corollary.** *For any natural number m, the assignment* $X \longmapsto \Pi_m X$ *defines a functor* $\mathbf{Kan} \longrightarrow {}_m\mathbf{Gpd}$ *that is left adjoint to the inclusion functor* ${}_m\mathbf{Gpd} \hookrightarrow \mathbf{Kan}$.

7.26. **Proposition.** *For any natural number m and any simplicial sets* X *and* Y, *the two maps* $X \times Y \longrightarrow X$ *and* $X \times Y \longrightarrow Y$ *together induce an isomorphism*

$$\Pi_m(X \times Y) \xrightarrow{\sim} \Pi_m X \times \Pi_m Y.$$

7.27. **Proposition.** *For any natural number m and any family* $\{X_i\}$ *of numerically generated spaces, the inclusions* $X_i \hookrightarrow \coprod_i X_i$ *together induce an isomorphism*

$$\coprod_i \Pi_m(X_i) \cong \Pi_m \left(\coprod_i X_i \right).$$

8. The singular simplicial set

8.1. **Lemma.** *For any object* $J \in \Delta$, *order the set* $\Delta^1_J = \mathrm{Mor}(J, [1])$ *so that for any* $\sigma, \tau \colon J \longrightarrow [1]$, *one has* $\sigma < \tau$ *just in case there exists* $j \in J$ *such that*

$$\sigma(j) < \tau(j).$$

Then Δ^1_J *is totally ordered and contains a minimum and maximum element, and for any morphism* $K \longrightarrow J$ *in* Δ, *the induced map*

$$\Delta^1_J \longrightarrow \Delta^1_K$$

preserves the order and minimum and maximum elements.

8.2. **Definition.** Define a functor

$$\Delta^\bullet_{\mathrm{top}} \colon \Delta \longrightarrow \mathbf{Num}$$

as follows: for any object $J \in \Delta$, let

$$\Delta^J_{\mathrm{top}} \subset \mathrm{Map}((\Delta^1_J)^\delta, I)$$

be the subspace consisting of those maps that preserve the order and minimum and maximum elements.

Now for any numerically generated space X, the *singular simplicial set* or *Poincaré* ∞*-groupoid* $\Pi_\infty(X)$ is the simplicial set defined by the formula

$$\Pi_\infty(X)_J := \mathrm{Map}(\Delta^J_{\mathrm{top}}, X).$$

This defines a functor

$$\Pi_\infty \colon \mathbf{Num} \longrightarrow s\mathbf{Set}.$$

8.3. **Theorem.** *For any numerically generated space* X, *the simplicial set* $\Pi_\infty(X)$ *is, in fact, an* ∞*-groupoid.*

8.4. **Theorem.** *Two maps* $\phi, \psi \colon X \longrightarrow Y$ *of numerically generated spaces are homotopic if and only if the corresponding morphisms*

$$\Pi_\infty(\phi), \Pi_\infty(\psi) \colon \Pi_\infty X \longrightarrow \Pi_\infty Y$$

are homotopic.

8.5. **Theorem.** *For any numerically generated space* X*, there is a natural bijection*

$$\pi_0 X \cong \pi_0 \Pi_\infty(X).$$

8.6. **Theorem.** *For any numerically generated space* X*, there is a natural equivalence of groupoids*

$$\Pi_1 X \simeq \Pi_1 \Pi_\infty(X) \simeq N\Pi_1 \Pi_\infty(X).$$

8.7. **Definition.** For any integer $m \geq 2$ and any numerically generated space X, write $\Pi_m(X)$ for the m-groupoid $\Pi_m \Pi_\infty(X)$.

8.8. **Definition.** For any simplicial set X, let $\sim$ be the equivalence relation on the coproduct

$$\coprod_{n \geq 0} (X_n^\delta \times \Delta_{top}^n)$$

generated by declaring that for any morphism $\phi \colon [m] \longrightarrow [n]$ of Δ and for any $(\sigma, x) \in X_n^\delta \times \Delta_{top}^m$, one has

$$(X(\phi)(\sigma), x) \sim (\sigma, \Delta_{top}^\bullet(\phi)(x)).$$

The *geometric realization* of X is the (numerically generated) quotient space

$$X_{top} := \left(\coprod_{n \geq 0} (X_n^\delta \times \Delta_{top}^n) \right) / \sim .$$

This defines a functor $(\cdot)_{top} \colon s\mathbf{Set} \longrightarrow \mathbf{Num}$.

8.9. **Proposition.** *The geometric realization functor* $(\cdot)_{top}$ *is left adjoint to the Poincaré* ∞*-groupoid functor* Π_∞*; that is, for any simplicial set* X *and any numerically generated space* Y*, there is a natural bijection*

$$\mathrm{Map}(X_{top}, Y) \cong \mathrm{Mor}(X, \Pi_\infty(Y)).$$