

# FROM OPERATOR CATEGORIES TO TOPOLOGICAL OPERADS

CLARK BARWICK

ABSTRACT. In this paper it is shown that an assortment of operads near to many topologists' hearts enjoy (homotopy) universal properties of an expressly combinatorial nature. These include the operads  $A_n$  and  $E_n$ . The main idea that makes this possible is the new notion of an *operator category*, which controls the homotopy types of these operads in a strong sense.

## CONTENTS

0. Introduction	1
1. Operator categories	5
2. Complete Segal operads	7
3. Wreath products	11
4. Boardman–Vogt tensor products and weak algebras	15
5. Perfect operator categories	18
6. The canonical monad on a perfect operator category	21
7. Leinster categories	25
8. Quasioperads and their algebras	27
9. Boardman–Vogt tensor products and $\infty$ -algebras	30
10. Complete Segal operads versus quasioperads	32
11. Some examples of weak $\Phi$ -monoids	39
Appendix A. A proof of Th. 4.12	41
Appendix B. A proof of Th. 6.9	44
Appendix C. A proof of Th. 6.15	47
References	49

## 0. INTRODUCTION

In this paper it is shown that an assortment of operads near to many topologists' hearts enjoy (homotopy) universal properties of an expressly combinatorial nature. These include the operads  $A_n$  ( $1 \leq n \leq \infty$ ) of associahedra of Stasheff [20], the little  $n$ -cubes operads  $E_n$  ( $1 \leq n \leq \infty$ ) of May [14].

In the limiting case, these results are well-known: an  $E_\infty$  operad is weakly equivalent to the terminal symmetric operad, and an  $A_\infty$  operad is weakly equivalent to the symmetrization of the terminal nonsymmetric operad. Our work here thus amounts to an extension of these observations to describe two infinite classes of operads. This extension may come as something of a surprise, as the homotopy types of the spaces that appear in the operads  $A_n$  and  $E_n$  are relatively intricate. Our

results here imply that these homotopy types are in a sense generated by simple combinatorial gadgets.

These universal properties are obtained by means of two successive generalizations of the ordinary homotopy theory of (colored) operads.

- First, we pass to the homotopy theory of *weak operads*, or  $\infty$ -*operads*. This is a variant of the theory of operads in which all the operadic compositions are weakened, so that they are specified (and associative) only up to coherent homotopy. There are many ways in which this weakening can be codified. The Moerdijk–Weiss theory of dendroidal sets provides one [16]; Lurie’s theory of  $\infty$ -operads provides another [13, Ch. 2]. The benefits of this approach are by now well-documented.
- Next, we *relativize* the theory of operads to allow new sorts of operads whose compositions are determined by more general combinatorics. These combinatorics are controlled by a category called an *operator category*, the objects of which are finite sets equipped with some additional structure. We obtain a theory of operads *over* a given operator category, in which an operation from a set of objects  $\{x_i\}$  to an object  $y$  can only be specified once the set  $\{x_i\}$  is endowed with the additional structure encoded in the operator category. For example, an operad over the operator category  $\mathbf{F}$  of finite sets is a symmetric operad; an operad over the operator category  $\mathbf{O}$  of totally ordered finite sets is a nonsymmetric operad. The one-point category  $\{1\}$  is an operator category as well; operads over  $\{1\}$  are simply categories.

Now any weak operad over any operator category admits a *symmetrization*, which is a symmetric operad obtained by freely adding all the necessary compositions. The main results of this paper (§11) state that for each of the operads  $E_n$  ( $1 \leq n \leq \infty$ ) and  $A_n$  ( $1 \leq n \leq \infty$ ), there is an operator category  $\Phi$  such that the operad is the symmetrization of the terminal weak operad over  $\Phi$ . As a result, the homotopy theory of algebras over these operads is completely controlled by the combinatorics of  $\Phi$ .

The theory of operator categories is really the new advance of this paper; the applications we describe here are only a first step. We introduce the  $\infty$ -category of operator categories in §1. Given an operator category  $\Phi$ , we develop the homotopy theory of weak operads in two ways. The first and simplest of these models (§2) is as suitable families of spaces over the nerve of a tree-like category  $\Delta_\Phi^{\text{op}}$  of finite sequences of objects of  $\Phi$  that is related to a version of the dendroidal category with level structure when  $\Phi = \mathbf{F}$ ; this homotopy theory — that of *complete Segal  $\Phi$ -operads* — can be described for any operator category and is easy to describe without many additional combinatorial complications. The second, a generalization of Lurie’s theory of  $\infty$ -operads, requires more technology, and will come later (§8).

One easy way to form new operator categories from old is *truncation* (Ex. 1.5). For any operator category  $\Phi$  and any integer  $n \geq 1$ , the full subcategory  $\Phi_{\leq n}$  spanned by those objects whose underlying finite set contains no more than  $n$  elements is an operator category. Applying this truncation to the operator category  $\mathbf{O}$ , we obtain the following without difficulty.

**Theorem.** *The operads  $A_n$  for  $1 \leq n < \infty$  are the symmetrizations of the terminal weak operad over  $\mathbf{O}_{\leq n}$ .*

There is a monoidal structure on the  $\infty$ -category of all operator categories called the *wreath product*, which we describe in §3. An object of the wreath product  $\Psi \wr \Phi$  is a pair consisting of an object  $I \in \Phi$  and a collection of objects  $\{J_i \mid i \in I\}$  indexed by the elements of  $I$ . The Boardman–Vogt tensor product of operads [8] is externalized relative to this wreath product in §4, so that the homotopy theory of algebras over a weak  $\Psi$ -operad  $C$  in the homotopy theory of algebras over a weak  $\Phi$ -operad  $D$  can be identified with the homotopy theory of algebras over a weak  $\Psi \wr \Phi$ -operad  $C \otimes D$ . In particular, the Boardman–Vogt tensor product of the terminal weak  $\Psi$ -operad with the terminal weak  $\Phi$ -operad is the terminal  $\Psi \wr \Phi$ -operad (Th. 4.12). If we form iterated wreath products

$$\mathbf{O}^{(n)} := \mathbf{O} \wr \mathbf{O} \wr \cdots \wr \mathbf{O}$$

of the operator category  $\mathbf{O}$ , we may therefore formulate the following.

**Theorem.** *The operads  $E_n$  for  $0 \leq n < \infty$  are the symmetrizations of the terminal weak operad over  $\mathbf{O}^{(n)}$ .*

This result identifies the homotopy theory of  $E_n$  algebras in any  $\mathbf{O}^{(n)}$ -operad  $C$  with the homotopy theory of  $A_\infty$ -algebras in  $A_\infty$ -algebras in  $\dots$  in  $A_\infty$ -algebras in  $C$ . This can be regarded as a homotopy-theoretic generalization of the *Eckmann–Hilton argument*, which identifies the category of monoids in monoids (in the category of sets, say) with the category of commutative monoids.

To prove this, we pass to a different way of describing weak operads and their algebras, which will only work for a special class of operator categories that we call *perfect* (§5). In effect, a perfect operator category  $\Phi$  admits a universal way of adding elements to any object. This defines a monad on  $\Phi$  (§6), and the free algebras for this monad form a supplementary category  $\Lambda(\Phi)$ , called the *Leinster category* (§7), which provides an alternative way of parametrizing the operations of an operad over  $\Phi$ . In many examples, it turns out that this Leinster category is in fact quite familiar (Ex. 7.5, 7.6, and 7.8):

$\Phi$	$\Lambda(\Phi)$
$\{1\}$	$\{1\}$
$\mathbf{O}$	$\Delta^{\text{op}}$
$\mathbf{F}$	$\Gamma^{\text{op}}$
$\mathbf{O}^{(n)}$	$\Theta_n^{\text{op}}$

(The last of these is Joyal’s category of disks.) Our second model of the homotopy theory of weak operads over a perfect operator category — that of  $\Phi$ -*quasioperads* — is a natural generalization of Lurie’s theory of  $\infty$ -operads (§8). In effect, a  $\Phi$ -quasioperad is an inner fibration  $X^\otimes \rightarrow N\Lambda(\Phi)$  enjoying certain properties analogous to the ones developed by Lurie. We show that this and our original model of the homotopy theory of weak operads over  $\Phi$  are equivalent in manner compatible with changes of operator category and the formation of  $\infty$ -categories of algebras (§10).

Now our characterization of the operads  $E_n$  essentially follows directly from results of Lurie [13]. This result is closely related to the results of Ayala and Hepworth [2].

**Example.** As an illustration of all this, let us consider the symmetric monoidal  $\infty$ -category of spectra. This may be represented in our first model (§2) as an  $\infty$ -category  $\mathbf{Sp}$  over the nerve  $N\Delta_{\mathbf{F}}^{\text{op}}$ . Our results yield the following identifications.

- The data of an  $A_n$ -ring spectrum ( $1 \leq n < \infty$ ) is equivalent to the data of a functor  $N\Delta_{\mathbf{O}_{\leq n}}^{\text{op}} \rightarrow \mathbf{Sp}$  lying over  $N\Delta_{\mathbf{F}}^{\text{op}}$ .
- The data of an  $A_\infty$ -ring spectrum is equivalent to the data of a functor  $N\Delta_{\mathbf{O}}^{\text{op}} \rightarrow \mathbf{Sp}$  lying over  $N\Delta_{\mathbf{F}}^{\text{op}}$ .
- The data of an  $E_n$ -ring spectrum ( $0 \leq n < \infty$ ) is equivalent to the data of a functor  $N\Delta_{\mathbf{O}^{(n)}}^{\text{op}} \rightarrow \mathbf{Sp}$  lying over  $N\Delta_{\mathbf{F}}^{\text{op}}$ .
- The data of an  $E_\infty$ -ring spectrum is equivalent to the data of a section of the functor  $\mathbf{Sp} \rightarrow N\Delta_{\mathbf{F}}^{\text{op}}$ .

Using our second model (§8), we may also regard the symmetric monoidal  $\infty$ -category of spectra as a particular  $\infty$ -category  $\mathbf{Sp}^\otimes$  lying over the nerve  $N\Gamma^{\text{op}}$ . Our results also yield the following identifications.

- The data of an  $E_\infty$  ring spectrum is then equivalent to the data of a section of the functor  $\mathbf{Sp}^\otimes \rightarrow N\Gamma^{\text{op}}$  that carries certain combinatorially defined morphisms (called *inert*) to cocartesian edges.
- The data of an  $E_n$  ring spectrum ( $0 \leq n < \infty$ ) is equivalent to the data of a functor  $N\Lambda(\mathbf{O}^{(n)}) \simeq N\Theta_n^{\text{op}} \rightarrow \mathbf{Sp}^\otimes$  over  $N\Gamma^{\text{op}}$  that carries inert morphisms to cocartesian edges.

This makes it clear that  $A_n$ - and  $E_n$ -ring spectra are fundamentally combinatorial objects, and their combinatorics are completely controlled by the relevant operator category. Of course there’s nothing special about spectra in this story; the same is true for any symmetric monoidal  $\infty$ -category.

The theory of operator categories also promises new horizons in the homotopy theory of operads. For example, we describe (Ex. 11.5) a new bifiltration on the  $E_\infty$  operad that seems to include much that is already known about obstruction theories for finding  $E_\infty$  structures on spectra. Additionally, there are many more exotic operator categories out there: the category of planar level trees (Ex. 1.7), the category of finite cyclically ordered sets (Ex. 1.8), and the category of finite simple graphs (Ex. 1.9) are all examples of operator categories.

**An apology.** The results of this paper largely date from 2005–2006. Since that time, there has been a series of dramatic advances in the study of homotopy coherent algebraic structures, spearheaded by Moerdijk–Weiss and Lurie. As soon as I managed to record some of these results, the techniques I employed had, discouragingly, become outmoded. It was some time before it became clear to me how the work here interacts with some of these new advances — particularly with Lurie’s framework. Ultimately, these advances have simplified the work here greatly; the main result of this paper is now an immediate consequence of the comparison between our theory of weak operads and Lurie’s. Nevertheless, my efforts to bring this work in line with current technology have led to an embarrassing delay in the publication of this work. I apologize for this, especially to the students who have sought to employ aspects of the theory introduced here.

**Acknowledgments.** All of this was inspired by Bertrand Toën’s visionary preprint [23]. Early conversations with Markus Spitzweck were instrumental to my understanding. It was an offhand remark in a paper of Tom Leinster [11, pp. 40–43] that led me to formulate the notion of perfection for operator categories; the *Leinster category* of §7 is thus named after him. Haynes Miller has always been very kind in his support and encouragement, and he has asked a number of questions that

helped refine my understanding of these objects. More recently, conversations with Chris Schommer-Pries revitalized my interest in this material; note that Lm. 8.3 in this paper is due to him. A visit from Clemens Berger helped me understand better how the work here interacts with concepts he's been developing since the dawn of the new millennium. In addition, I've benefitted from conversations with David Ayala, Ezra Getzler, John Rognes, and Sarah Whitehouse.

## 1. OPERATOR CATEGORIES

The objects of an operator category are finite sets equipped with some additional structure. Such an object will be regarded as an indexing set for some multiplication law. The structure of the operator category can thus be thought of determining the associativity and commutativity constraints on that law.

**1.1. Notation.** For any ordinary category  $\Phi$  with a terminal object  $1$  and for any object  $K \in \Phi$ , we write  $|K| := \text{Mor}_\Phi(1, K)$ . For any  $i \in |K|$ , it will be convenient to denote the morphism  $i: 1 \rightarrow K$  as  $\{i\} \hookrightarrow K$ .

**1.2. Definition.** An *operator category*  $\Phi$  is an essentially small category that satisfies the following three conditions.

(1.2.1) The category  $\Phi$  has a terminal object.

(1.2.2) For any morphism  $J \rightarrow I$  of  $\Phi$  and for any point  $i \in |I|$ , there exists a fiber  $J_i := \{i\} \times_I J$ .

(1.2.3) For any pair of objects  $I, J \in \Phi$ , the set  $\text{Mor}_\Phi(J, I)$  is finite.

**1.3 (A note on terminology).** The notion we have defined here is distinct from the notion of a *category of operators* used in the brilliant work of May and Thomason [15], and it serves a distinct mathematical role. We hope that the obvious similarity in nomenclature will not lead to confusion.

There are very many interesting examples of operator categories, but for now, let us focus on a small number of these.

**1.4. Example.** The following categories are operator categories:

(1.4.1) the trivial category  $\{1\}$ ;

(1.4.2) the category  $\mathbf{O}$  of ordered finite sets; and

(1.4.3) the category  $\mathbf{F}$  of finite sets.

**1.5. Example.** For any operator category  $\Phi$  and for any integer  $n \geq 1$ , write  $\Phi_{\leq n}$  for the full subcategory of  $\Phi$  spanned by those objects  $I \in \Phi$  such that  $\#|I| \leq n$ . Then the category  $\Phi_{\leq n}$  is an operator category as well.

**1.6. Example.** Suppose  $\Psi$  and  $\Phi$  two operator categories. Then we may define a category  $\Psi \wr \Phi$  as follows. An object of  $\Psi \wr \Phi$  will be a pair  $(I, M)$  consisting of an object  $I \in \Phi$  and a collection  $M = \{M_i\}_{i \in |I|}$  of objects of  $\Psi$ , indexed by the points of  $I$ . A morphism  $(\eta, \omega): (J, N) \rightarrow (I, M)$  of  $\Psi \wr \Phi$  consists of a morphism  $\eta: J \rightarrow I$  of  $\Phi$  and a collection

$$\{\omega_j: N_j \rightarrow M_{\eta(j)}\}_{j \in |J|}$$

of morphisms of  $\Psi$ , indexed by the points of  $J$ . Then  $\Psi \wr \Phi$  is an operator category. In §3 we will give a more systematic discussion of this story, and we will show that this *wreath product* of operator categories in fact determines a monoidal structure on the collection of all operator categories.

1.7. **Example.** Suppose  $\Phi$  any operator category. Then we may define a *semidirect product* category  $\Phi \rtimes \mathbf{O}$  as follows. An object of  $\Phi \rtimes \mathbf{O}$  will be a pair  $(I, M)$  consisting of a totally ordered finite set  $I \in \mathbf{O}$  and a functor  $M: I \rightarrow \Phi$ . A morphism  $(\eta, \omega): (J, N) \rightarrow (I, M)$  of  $\Phi \rtimes \mathbf{O}$  consists of a morphism  $\eta: J \rightarrow I$  of  $\mathbf{O}$  and a natural transformation

$$\omega: N \rightarrow \eta^* M$$

of functors  $J \rightarrow \Phi$ . Then  $\Phi \rtimes \mathbf{O}$  is an operator category.

1.8. **Example.** Denote by  $\mathbf{C}$  the category of finite cyclically ordered sets and monotone maps, i.e., maps  $\phi: J \rightarrow I$  such that for any  $r, s, t \in J$ , if  $[\phi(r), \phi(s), \phi(t)]$  in  $I$ , then  $[r, s, t]$  in  $J$ . Then  $\mathbf{C}$  is an operator category.

1.9. **Example.** Denote by  $\mathbf{G}$  the category of finite simple graphs. Then  $\mathbf{G}$  is an operator category.

1.10. **Definition.** A functor  $G: \Psi \rightarrow \Phi$  between operator categories will be said to be *admissible* if it preserves terminal objects and the formation of fibers. An admissible functor  $G: \Psi \rightarrow \Phi$  will be said to be an *operator morphism* if in addition, for any object  $I$  of  $\Psi$ , the induced morphism  $|I| \rightarrow |GI|$  is a surjection.

1.11. **Example.** (1.11.1) Any equivalence  $\Psi \xrightarrow{\sim} \Phi$  between operator categories is an operator morphism.

(1.11.2) For any operator category  $\Phi$ , the assignment  $I \mapsto |I|$  is an operator morphism  $u: \Phi \rightarrow \mathbf{F}$  is an operator morphism.

(1.11.3) For any operator category  $\Phi$ , the inclusion  $\{1\} \hookrightarrow \Phi$  of the terminal object is an operator morphism.

(1.11.4) For any operator category  $\Phi$  and for any positive integer  $n$ , the inclusion  $\Phi_{\leq n} \hookrightarrow \Phi$  is an operator morphism.

(1.11.5) For any operator category  $\Phi$ , the unique functor  $\Phi \rightarrow \{1\}$  is an admissible functor, but it is generally not an operator morphism.

By considering the fibers over distinct points of an object, one proves the following.

1.12. **Proposition.** *Suppose  $G: \Psi \rightarrow \Phi$  an operator morphism. Then for any object  $I$  of  $\Psi$ , the induced morphism  $|I| \rightarrow |GI|$  is a bijection.*

1.12.1. **Corollary.** *Suppose*

$$\begin{array}{ccc} & \Psi & \\ G \nearrow & & \searrow F \\ X & \xrightarrow{H} & \Phi \end{array}$$

*a commutative triangle of admissible functors in which  $F$  is an operator morphism; then  $G$  is an operator morphism if and only if  $H$  is.*

We organize the collection of operator categories into an  $\infty$ -category.

1.13. **Notation.** Denote by  $\mathbf{Adm}$  the (strict) 2-category in which the objects are small operator categories, the 1-morphisms are admissible functors, and the 2-morphisms are isomorphisms of functors. Denote by  $\mathbf{Op}$  the sub-2-category of  $\mathbf{Adm}$

in which the objects are small operator categories, the 1-morphisms are operator morphisms, and the 2-morphisms are isomorphisms of functors.

Applying the nerve to each Mor-groupoid in  $\mathbf{Adm}$  and  $\mathbf{Op}$ , we obtain categories enriched in fibrant simplicial sets, and we may apply the simplicial nerve to obtain  $\infty$ -categories that are 2-categories in the sense of [12, §2.3.4] (which could perhaps more precisely be called “(2, 1)-categories”). We will refer to these  $\infty$ -categories as  $\mathbf{Adm}$  and  $\mathbf{Op}$ .

As a result of Pr. 1.12, we have the following.

**1.14. Proposition.** *The trivial operator category  $\{1\}$  is initial in both  $\mathbf{Adm}$  and  $\mathbf{Op}$ , and it is terminal in  $\mathbf{Adm}$ . The operator category  $\mathbf{F}$  is terminal in  $\mathbf{Op}$ .*

## 2. COMPLETE SEGAL OPERADS

Any operator category gives rise to a theory of operads (elsewhere called a colored operad or multicategory). Here we define a *weak* version of this theory, as well as its theory of algebras. To this end, we first single out an important class of morphisms of an operator category.

**2.1. Definition.** A morphism  $K \rightarrow J$  of an operator category  $\Phi$  is a ***fiber inclusion*** if there exists a morphism  $J \rightarrow I$  and a point  $i \in |I|$  such that the square

$$\begin{array}{ccc} K & \longrightarrow & J \\ \downarrow & & \downarrow \\ \{i\} & \hookrightarrow & I \end{array}$$

is a pullback square. A morphism  $K \rightarrow J$  is an ***interval inclusion*** if it can be written as the composite of a finite sequence of fiber inclusions. An interval inclusion will be denoted by a hooked arrow  $K \hookrightarrow J$ .

**2.2. Example.** (2.2.1) Any isomorphism of an operator category  $\Phi$  is a fiber inclusion, and any point of any object of an operator category  $\Phi$  is a fiber inclusion.

(2.2.2) Suppose  $I$  an object of  $\mathbf{O}$ . Then for any elements  $i_0, i_1 \in |I|$ , write  $[i_0, i_1]$  for the subset of elements  $i \in |I|$  such that  $i_0 \leq i \leq i_1$ , with the induced ordering. A morphism  $J \rightarrow I$  is a fiber inclusion if and only if it is a monomorphism whose image is precisely  $[i_0, i_1]$  for some elements  $i_0, i_1 \in |I|$ . Interval inclusions and fibers inclusions coincide in  $\mathbf{O}$ .

(2.2.3) In the category  $\mathbf{F}$ , fiber inclusions are precisely monomorphisms. Again interval inclusions and fibers inclusions coincide.

**2.3.** We make the following observations.

(2.3.1) Interval inclusions in an operator category are monomorphisms.

(2.3.2) If  $\Phi$  is an operator category, then the pullback of any morphism  $L \rightarrow J$  along any interval inclusion  $K \hookrightarrow J$  exists, and the canonical morphism  $K \times_J L \rightarrow L$  is an interval inclusion.

(2.3.3) Suppose  $\Phi$  an operator category; suppose  $\phi: K \hookrightarrow J$  an interval inclusion in  $\Phi$ ; and suppose  $\psi: L \rightarrow K$  a morphism of  $\Phi$ . Then  $\psi$  is an interval inclusion if and only if  $\phi \circ \psi$  is.

(2.3.4) Suppose  $\Phi$  an operator category,  $K \hookrightarrow J$  an interval inclusion, and  $j \in |J|$  a point. Then either the fiber  $K_j$  is a terminal object or else  $|K_j| = \emptyset$ .

**2.4. Definition.** Suppose  $\Phi$  an operator category. Then a  $\Phi$ -*sequence* is a pair  $(\mathbf{m}, I)$  consisting of an object  $m \in \Delta$  and a functor  $I: \mathbf{m} \rightarrow \Phi$ . We will denote such an object by

$$[I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_m].$$

A *morphism*  $(\eta, \phi): (\mathbf{n}, J) \rightarrow (\mathbf{m}, I)$  of  $\Phi$ -sequences consists of a morphism  $\eta: \mathbf{n} \rightarrow \mathbf{m}$  of  $\Delta$  and a natural transformation  $\phi: J \rightarrow I \circ \eta$  such that for any integer  $0 \leq k \leq m$ , the morphism  $\phi_k: J_k \rightarrow I_{\eta(k)}$  is an interval inclusion, and for any pair of integers  $0 \leq k \leq \ell \leq m$ , the square

$$\begin{array}{ccc} J_k & \xrightarrow{\phi_k} & I_{\eta(k)} \\ \downarrow & & \downarrow \\ J_\ell & \xrightarrow{\phi_\ell} & I_{\eta(\ell)} \end{array}$$

is a pullback of  $\Phi$ . Denote by  $\Delta_\Phi$  the category of  $\Phi$ -sequences.

Any admissible functor  $G: \Psi \rightarrow \Phi$  induces a functor  $\Delta_\Psi \rightarrow \Delta_\Phi$  given by  $(\mathbf{m}, I) \mapsto (\mathbf{m}, G \circ I)$ .

**2.5. Notation.** Suppose  $\Phi$  an operator category, and suppose  $X: \Delta_\Phi^{\text{op}} \rightarrow \mathbf{Kan}$  a functor. We study three classes of maps.

(2.5.1) For any  $\Phi$ -sequence  $(m, I)$  and for any point  $i \in |I_m|$ , one obtains a map

$$X[I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_m] \rightarrow X[I_{0,i} \rightarrow I_{1,i} \rightarrow \cdots \rightarrow \{i\}].$$

Consequently, one obtains a map

$$p_{(\mathbf{m}, I)}: X[I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_m] \rightarrow \prod_{i \in |I_m|} X[I_{0,i} \rightarrow I_{1,i} \rightarrow \cdots \rightarrow \{i\}].$$

(2.5.2) For any  $\Phi$ -sequence  $(\mathbf{m}, I)$ , one obtains a map

$$s_{\mathbf{m}, I}: X[I_0 \rightarrow \cdots \rightarrow I_m] \rightarrow X[I_0 \rightarrow I_1] \times_{X[I_1]}^h \cdots \times_{X[I_{m-1}]}^h X[I_{m-1} \rightarrow I_m],$$

where the target is the homotopy fiber product.

(2.5.3) Lastly, the inclusion  $\{1\} \hookrightarrow \Phi$  induces an inclusion  $\Delta \cong \Delta_{\{1\}} \hookrightarrow \Delta_\Phi$ ; hence  $X$  restricts to a simplicial space  $(X|\Delta^{\text{op}})$ , and one can define the map

$$r: (X|\Delta^{\text{op}})_0 \rightarrow (X|\Delta^{\text{op}})_K,$$

where  $K = \Delta^3 / (\Delta^{\{0,2\}} \sqcup \Delta^{\{1,3\}})$ , and  $(X|\Delta^{\text{op}})_K$  is the homotopy limit of the diagram

$$\Delta_{/K}^{\text{op}} \rightarrow \Delta^{\text{op}} \xrightarrow{(X|\Delta^{\text{op}})} \mathbf{sSet}.$$

**2.6. Definition.** Suppose  $\Phi$  an operator category. A *complete Segal  $\Phi$ -operad* is a left fibration  $q: X \rightarrow N\Delta_\Phi^{\text{op}}$  such that any functor  $\Delta_\Phi^{\text{op}} \rightarrow \mathbf{Kan}$  that classifies  $q$  has the property that the maps  $p_{(\mathbf{m}, I)}$ ,  $s_{(\mathbf{m}, I)}$ , and  $r$  (Nt. 2.5) are all equivalences. A *morphism  $g: X \rightarrow Y$  of complete Segal  $\Phi$ -operads* is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ & \searrow & \swarrow \\ & N\Delta_\Phi^{\text{op}} & \end{array}$$

A morphism  $g: X \rightarrow Y$  will be said to be a **equivalence of complete Segal  $\Phi$ -operads** if it is a covariant weak equivalence.

Denote by  $\mathbf{Operad}_{\text{CSS}}^{\Phi, \Delta}$  the full simplicial subcategory of  $s\mathbf{Set}/N\Delta_{\Phi}^{\text{op}}$  spanned by the weak  $\Phi$ -operads. This is a fibrant simplicial category, so we can define an  $\infty$ -category  $\mathbf{Operad}_{\text{CSS}}^{\Phi}$  as the simplicial nerve of  $\mathbf{Operad}_{\text{CSS}}^{\Phi}$ .

2.7. Suppose  $\Phi$  an operator category. We refer to the fiber of a complete Segal  $\Phi$ -operad  $X$  over an object

$$[I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_m] \in \Delta_{\Phi}^{\text{op}}$$

as the **space of operations of type**  $[I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_m]$ . The space of operations of type  $\{1\}$  is the **space of objects** of  $X$  or *interior  $\infty$ -groupoid* of  $X$ , and the space of operations of type  $I$  is the **space of  $I$ -tuples of objects** of  $X$ . Let us denote by

$$[I_0/I_1/\cdots/I_m] \text{Map}_X((x_{i_0}^0)_{i_0 \in |I_0|}, \dots, (x_{i_m}^m)_{i_m \in |I_m|})$$

the fiber of the map  $X[I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_m] \rightarrow X(I_0) \times \cdots \times X(I_m)$  over the vertex  $((x_{i_0}^0)_{i_0 \in |I_0|}, \dots, (x_{i_m}^m)_{i_m \in |I_m|})$ . The condition that  $X$  is a weak  $\Phi$ -operad then gives equivalences

$$\begin{aligned} [J/I/\{1\}] \text{Map}_X((x_j)_{j \in |J|}, (y_i)_{i \in |I|}, z) \\ \simeq [I/\{1\}] \text{Map}_X((y_i)_{i \in |I|}, z) \times [J/I] \text{Map}_X((x_j)_{j \in |J|}, (y_i)_{i \in |I|}) \\ \simeq [I/\{1\}] \text{Map}_X((y_i)_{i \in |I|}, z) \times \prod_{i \in |I|} [J_i/\{i\}] \text{Map}_X((x_j)_{j \in |J_i|}, y_i), \end{aligned}$$

and the map

$$[J/I/\{1\}] \text{Map}_X((x_j)_{j \in |J|}, (y_i)_{i \in |I|}, z) \rightarrow [J/\{1\}] \text{Map}_X((x_j)_{j \in |J|}, z)$$

induced by the map

$$\begin{array}{ccc} J & \longrightarrow & \{1\} \\ \parallel & & \parallel \\ J & \longrightarrow & I \longrightarrow \{1\} \end{array}$$

amounts to a **polycomposition map**, which is defined up to coherent homotopy. The functoriality in  $N\Delta_{\Phi}^{\text{op}}$  amounts to a coherent associativity condition.

We immediately obtain the following characterization of equivalences between complete Segal  $\Phi$ -operads.

2.8. **Proposition.** *Suppose  $\Phi$  an operator category. Then a morphism  $g: X \rightarrow Y$  of complete Segal  $\Phi$ -operads is an equivalence just in case the following conditions are satisfied.*

(2.8.1) Essential surjectivity. *The map  $\pi_0 X[\{1\}] \rightarrow \pi_0 Y[\{1\}]$  is surjective.*

(2.8.2) Full faithfulness. *For any object  $I \in \Phi$ , any vertex  $x \in X[I]_0$ , and any vertex  $y \in X[\{1\}]_0$ , the induced map*

$$[I/\{1\}] \text{Map}_X(x, y) \rightarrow [I/\{1\}] \text{Map}_Y(g(x), g(y))$$

*is an equivalence.*

**2.9. Example.** For any operator category  $\Phi$ , the identity functor on the simplicial set  $N\Delta_{\Phi}^{\text{op}}$  is a complete Segal  $\Phi$ -operad — the *terminal complete Segal  $\Phi$ -operad*, which we denote  $U_{\Phi}$ . These complete Segal  $\Phi$ -operads, for suitable choice of  $\Phi$ , give rise to all the operads discussed in the introduction. In particular, when  $\Phi = \mathbf{F}$ , we show that the terminal weak  $\mathbf{F}$ -operad  $U_{\mathbf{F}}$  is equivalent to the operad  $E_{\infty}$  (§11).

**2.10. Example.** When  $\Phi = \{1\}$ , we find that a complete Segal  $\{1\}$ -operad is a left fibration  $X \rightarrow N\Delta^{\text{op}}$  classified by a complete Segal *space* in the sense of Rezk [17]. (In particular,  $\mathbf{Operad}_{\text{CSS}}^{\{1\}}$  is a homotopy theory of  $(\infty, 1)$ -categories, in the sense of [5].) For simplicity, we will, by a small abuse, simply call such left fibrations *complete Segal spaces*.

**2.11. Example.** When  $\Phi = \mathbf{F}$ , we obtain a new homotopy theory of weak symmetric operads, which we will show is equivalent to Lurie’s in §10.

Using the left Bousfield localization [9, 3] of the covariant model structure [12, Pr. 2.1.4.7] with respect to the set  $S_{\Phi}$  of those morphisms that represent the morphisms of 2.5, one obtains the following.

**2.12. Proposition.** *The category  $s\mathbf{Set}_{/N\Delta_{\Phi}^{\text{op}}}$  admits a left proper, tractable, simplicial model structure — called the **operadic model structure** — with the following properties.*

- (2.12.1) *A map  $X \rightarrow Y$  over  $N\Delta_{\Phi}^{\text{op}}$  is a cofibration just in case it is a monomorphism.*
- (2.12.2) *An object  $X \rightarrow N\Delta_{\Phi}^{\text{op}}$  is fibrant just in case it is a complete Segal  $\Phi$ -operad.*
- (2.12.3) *A map  $X \rightarrow Y$  of simplicial sets over  $N\Delta_{\Phi}^{\text{op}}$  is a weak equivalence if and only if, for any complete Segal  $\Phi$ -operad  $Z$ , the induced map*

$$\text{Map}_{/N\Delta_{\Phi}^{\text{op}}}(Y, Z) \rightarrow \text{Map}_{/N\Delta_{\Phi}^{\text{op}}}(X, Z)$$

*is a weak equivalence.*

- (2.12.4) *A map  $X \rightarrow Y$  between complete Segal  $\Phi$ -operads is a weak equivalence just in case it is an equivalence of complete Segal  $\Phi$ -operads.*

**2.12.1. Corollary.** *The  $\infty$ -category  $\mathbf{Operad}_{\text{CSS}}^{\Phi}$  of complete Segal  $\Phi$ -operads is an accessible localization of the functor  $\infty$ -category  $\text{Fun}(N\Delta_{\Phi}^{\text{op}}, \mathbf{Kan})$ ; in particular, it is a presentable  $\infty$ -category.*

Critically, the homotopy theory of weak operads over operator categories is functorial with respect to operator morphisms.

**2.13. Proposition.** *For any operator morphism  $G: \Psi \rightarrow \Phi$ , the adjunction*

$$G_!: s\mathbf{Set}_{/N\Delta_{\Psi}^{\text{op}}} \rightleftarrows s\mathbf{Set}_{/N\Delta_{\Phi}^{\text{op}}}: G^*$$

*is a Quillen adjunction for the operadic model structure.*

*Proof.* It is enough to note that the functor  $G^*$  is a right adjoint for the covariant model structure, and a left fibration  $X \rightarrow N\Delta_{\Phi}^{\text{op}}$  is a complete Segal  $\Phi$ -operad only if  $G^*X \cong X \times_{N\Delta_{\Phi}^{\text{op}}} N\Delta_{\Psi}^{\text{op}} \rightarrow N\Delta_{\Psi}^{\text{op}}$  is a complete Segal  $\Psi$ -operad.  $\square$

2.13.1. **Corollary.** *An operator morphism  $G: \Psi \rightarrow \Phi$  induces an adjunction*

$$G_!: \mathbf{Operad}_{\text{CSS}}^{\Psi} \rightleftarrows \mathbf{Operad}_{\text{CSS}}^{\Phi}: G^*$$

of  $\infty$ -categories.

2.14. **Example.** For any operator category  $\Psi$ , the right adjoint

$$p^*: \mathbf{Operad}^{\Psi} \rightarrow \mathbf{Operad}^{\{1\}}$$

induced by the inclusion  $p: \{1\} \hookrightarrow \Psi$  carries any complete Segal  $\Psi$ -operad to its *underlying complete Segal space*.

2.15. **Example.** When  $\Phi = \mathbf{F}$ , the notion of a complete Segal  $\Phi$ -operad is closely related to the usual notion of a symmetric operad in simplicial sets. For any operator category  $\Psi$ , the right adjoint  $u^*: \mathbf{Operad}_{\text{CSS}}^{\mathbf{F}} \rightarrow \mathbf{Operad}_{\text{CSS}}^{\Psi}$  induced by the essentially unique operator morphism  $u: \Psi \rightarrow \mathbf{F}$  carries any weak  $\mathbf{F}$ -operad to its *underlying weak  $\Psi$ -operad*. The left adjoint  $\mathbf{Symm}: u_!$  carries any weak  $\Psi$ -operad to its *symmetrization*.

2.16. **Example.** When  $\Psi = \mathbf{O}$ , we show below that the symmetrization of the terminal complete Segal  $\mathbf{O}$ -operad  $U_{\mathbf{O}}$  is equivalent to the operad  $E_1$ , or, equivalently, the operad  $A_{\infty}$ . When  $\Psi = \mathbf{O}_{\leq n}$  for some integer  $n \geq 1$ , we show below that the symmetrization of the terminal complete Segal  $\mathbf{O}_{\leq n}$ -operad  $U_{\mathbf{O}_{\leq n}}$  is equivalent to the operad  $A_n$  (§11).

We note that it will sometimes be convenient to work with a straightened variant of the operadic model structure of Pr. 2.12, which is provided by the following left Bousfield localization of the injective model structure.

2.17. **Proposition.** *The category  $\text{Fun}(\Delta_{\Phi}^{\text{op}}, s\mathbf{Set})$  admits a left proper tractable model structure — called the **operadic model structure** — with the following properties.*

- (2.17.1) *A natural transformation  $X \rightarrow Y$  is a cofibration just in case it is a monomorphism.*
- (2.17.2) *An object  $X$  is fibrant just in case it is valued in Kan complexes, and it classifies a complete Segal  $\Phi$ -operad.*
- (2.17.3) *A natural transformation  $X \rightarrow Y$  is a weak equivalence if and only if, for any fibrant object  $Z$ , the induced map*

$$\text{Map}(Y, Z) \rightarrow \text{Map}(X, Z)$$

*is a weak equivalence.*

2.18. For any operator category  $\Phi$ , it is clear that the straightening/unstraightening Quillen equivalence of [12, §2.2.1] localizes to a Quillen equivalence

$$\text{St}: s\mathbf{Set}_{/N\Delta^{\text{op}}} \rightleftarrows \text{Fun}(\Delta_{\Phi}^{\text{op}}, s\mathbf{Set}): \text{Un}$$

between the operadic model structures on  $\text{Fun}(\Delta_{\Phi}^{\text{op}}, s\mathbf{Set})$  and  $s\mathbf{Set}_{/N\Delta_{\Phi}^{\text{op}}}$ .

### 3. WREATH PRODUCTS

Roughly speaking, if the objects of an operator category index a certain sort of multiplications, then the objects of a wreath product of two operator categories  $\Phi$  and  $\Psi$  index  $\Phi$ -multiplications in objects that already possess  $\Psi$ -multiplications. With this insight, we can introduce the operator categories  $\mathbf{O}^{(n)}$  ( $1 \leq n < \infty$ ), which are key to the combinatorial gadgets that characterize the operads  $E_n$ .

**3.1. Definition.** Suppose  $\Phi$  an operator category. Then a *coronal fibration*

$$p: X \longrightarrow N\Phi$$

is a cartesian fibration such that, for any object  $I \in \Phi$ , the functors

$$\{X_I \longrightarrow X_{\{i\}} \mid i \in |I|\}$$

together exhibit the fiber  $X_I$  as a product of the fibers  $X_{\{i\}}$ . In this situation,  $p$  will be said to *exhibit  $X$  as a wreath product of  $X_1$  with  $\Phi$* .

**3.2. Notation.** Denote by  $\mathcal{O}^{\text{cart}}(\mathbf{Cat}_\infty)$  the subcategory [12, §1.2.11] of the  $\infty$ -category

$$\mathcal{O}(\mathbf{Cat}_\infty) := \text{Fun}(\Delta^1, \mathbf{Cat}_\infty)$$

whose objects are cartesian fibrations and whose morphisms carry cartesian morphisms to cartesian morphisms. Let  $\mathbf{Cat}_{\infty/S}^{\text{cart}}$  denote the fiber of the target functor

$$t: \mathcal{O}^{\text{cart}}(\mathbf{Cat}_\infty) \longrightarrow \mathbf{Cat}_\infty$$

over an object  $\{S\} \subset \mathbf{Cat}_\infty$ . Note that  $\mathbf{Cat}_{\infty/S}^{\text{cart}}$  may be identified with the nerve of the cartesian simplicial model category of marked simplicial sets over  $S$  [12, Pr. 3.1.3.7], whence it is the relative nerve of the category of cartesian fibrations over  $S$ , equipped with the cartesian equivalences.

Now for any operator category  $\Phi$ , denote by

$$\mathbf{Cor}_{/N\Phi} \subset \mathbf{Cat}_{\infty, /N\Phi}^{\text{cart}}$$

the full subcategory spanned by the coronal fibrations.

**3.3. Lemma.** *Suppose  $\Phi$  an operator category with a terminal object 1. Then the functor  $\mathbf{Cor}_{/N\Phi} \longrightarrow \mathbf{Cat}_\infty$  given by the extraction  $X \longmapsto X_1$  of the fiber over 1 is a trivial fibration.*

*Proof.* It is a direct consequence of straightening [12, §3.2] that the  $\infty$ -category  $\mathbf{Cat}_{\infty, /N\Phi}^{\text{cart}}$  is equivalent to the functor  $\infty$ -category  $\text{Fun}(N\Phi^{\text{op}}, \mathbf{Cat}_\infty)$ , and the functor  $\mathbf{Cat}_{\infty, /N\Phi}^{\text{cart}} \longrightarrow \mathbf{Cat}_\infty$  given by the assignment  $X \longmapsto X_1$  corresponds to evaluation at  $1 \in N\Phi^{\text{op}}$ . Now the result follows from the observation that a cartesian fibration  $p: X \longrightarrow N\Phi$  is a coronal fibration just in case the corresponding functor  $F_p: N\Phi^{\text{op}} \longrightarrow \mathbf{Cat}_\infty$  is a right Kan extension of the restriction of  $F_p$  to  $\{1\}$ .  $\square$

**3.4. Notation.** For any  $\infty$ -category  $C$  and any operator category  $\Phi$ , we write  $C \wr \Phi$  for the total space  $X$  of a coronal fibration  $p: X \longrightarrow N\Phi$  such that  $C$  appears as the fiber over a terminal object. One can make the assignment  $C \longrightarrow C \wr \Phi$  into a functor by choosing a terminal object  $1 \in \Phi$  and a section of the trivial fibration  $\mathbf{Cor}_{/N\Phi} \longrightarrow \mathbf{Cat}_\infty$  informally described as the assignment  $X \longmapsto X_1$ . The space of these choices is contractible.

**3.5. Lemma.** *If  $F: \Psi \longrightarrow \Phi$  is an operator morphism, then the pullback*

$$F^*p: X \times_{N\Phi} N\Psi \longrightarrow N\Psi$$

*of a coronal fibration  $p: X \longrightarrow N\Phi$  is a coronal fibration.*

*Proof.* Suppose  $I \in \Psi$ . The maps  $(X \times_{N\Phi} N\Psi)_I \longrightarrow (X \times_{N\Phi} N\Psi)_{\{i\}}$  for  $i \in |I|$  can be identified with the maps  $X_{FI} \longrightarrow X_{F\{i\}}$ . Since the map  $|I| \longrightarrow |FI|$  is a bijection, these maps exhibit the fiber  $(X \times_{N\Phi} N\Psi)_I$  as a product of the fibers  $(X \times_{N\Phi} N\Psi)_{\{i\}}$ .  $\square$

This lemma, combined with the previous one, implies that pullback along an operator morphism  $F: \Psi \rightarrow \Phi$  induces an equivalence of  $\infty$ -categories

$$\mathbf{Cor}/_{N\Phi} \xrightarrow{\simeq} \mathbf{Cor}/_{N\Psi};$$

they also imply that the whole  $\infty$ -category of coronal fibrations is equivalent to the  $\infty$ -category  $\mathbf{Op} \times \mathbf{Cat}_\infty$ . We will return momentarily to a special case of this observation.

**3.6. Lemma.** *Suppose  $\Phi$  an operator category, and suppose  $p: X \rightarrow N\Phi$  a coronal fibration such that the fiber  $X_1$  is the nerve of a operator category. Then  $X$  itself is also the nerve of an operator category, and  $p$  is the nerve of admissible functor.*

*Proof.* Since  $X_1$  is a 1-category, every fiber  $X_I \simeq X_1^{\times |I|}$  is a 1-category. It follows that  $X$  itself is a 1-category. The fact that  $X$  has a terminal object and all fibers follows directly from [12, Pr. 4.3.1.10].  $\square$

Informally, we conclude that the wreath product of two operator categories is again an operator category.

**3.7. Lemma.** *Suppose  $\Phi$  and  $\Phi'$  operator categories, and suppose  $p: \Phi' \rightarrow \Phi$  an admissible functor whose nerve is a coronal fibration. Then for any operator morphism  $F: \Psi \rightarrow \Phi$  and any pullback diagram*

$$\begin{array}{ccc} \Psi' & \xrightarrow{F'} & \Phi' \\ q \downarrow & & \downarrow p \\ \Psi & \xrightarrow{F} & \Phi, \end{array}$$

*the functor  $q$  is admissible, and the functor  $F'$  is an operator morphism.*

*Proof.* It is a simple matter to see that  $q$  and  $F'$  are admissible functors. To check that  $F'$  is an operator morphism, let us note first that  $F'$  induces an equivalence  $\Psi'_* \xrightarrow{\simeq} \Phi'_*$ ; hence for any object  $K \in \Psi'_*$ , the natural map  $|K| \rightarrow |F'K|$  is a bijection. Now for any object  $J \in \Psi$ , the decomposition  $\Psi'_J \simeq (\Psi'_*)^{\times |J|}$  gives, for any object  $K$  of  $\Psi'$  lying over  $J$ , a decomposition of finite sets

$$|K| \cong \prod_{j \in |J|} |K_{\{j\}}|.$$

After applying  $F'$ , we similarly obtain a decomposition of finite sets

$$|F'K| \cong \prod_{i \in |FJ|} |(F'K)_{\{i\}}|.$$

We thus conclude that the map  $|K| \rightarrow |F'K|$  is a bijection since both the map  $|J| \rightarrow |FJ|$  and all the maps  $|K_{\{j\}}| \rightarrow |(F'K)_{\{i\}}|$  are bijections.  $\square$

We now set about showing that the wreath product defines a monoidal structure on the  $\infty$ -category  $\mathbf{Op}$ .

**3.8. Notation.** Denote by  $\mathbf{M}$  the ordinary category whose objects are pairs  $(m, i)$  consisting of an integer  $m \geq 0$  and an integer  $0 \leq i \leq m$  and whose morphisms  $(n, j) \rightarrow (m, i)$  are maps  $\phi: \mathbf{m} \rightarrow \mathbf{n}$  of  $\Delta$  such that  $j \leq \phi(i)$ . This category comes equipped with a natural projection  $\mathbf{M} \rightarrow \Delta^{\text{op}}$ .

Denote by  $E(\mathbf{Adm})$  the simplicial set over  $N\Delta$  specified by the following universal property. We require, for any simplicial set  $K$  and any map  $\sigma: K \rightarrow N\Delta^{\text{op}}$ , a bijection

$$\text{Mor}_{/N\Delta^{\text{op}}}(K, E(\mathbf{Adm})^{\text{op}}) \cong \text{Mor}(K \times_{N\Delta^{\text{op}}} N\mathbf{M}, \mathbf{Adm}^{\text{op}}),$$

functorial in  $\sigma$ . Now by [12, Cor. 3.2.2.13], the map  $E(\mathbf{Adm}) \rightarrow N\Delta$  is a cartesian fibration, and  $E(\mathbf{Adm})$  is an  $\infty$ -category whose objects are pairs  $(m, X)$  consisting of an integer  $m \geq 0$  and a functor  $X: (\Delta^m)^{\text{op}} \rightarrow \mathbf{Adm}$ . A morphism  $(\phi, g): (n, Y) \rightarrow (m, X)$  can be regarded as a map  $\phi: \mathbf{n} \rightarrow \mathbf{m}$  of  $\Delta$  and an edge  $g: Y \rightarrow \phi^*X$  in  $\text{Fun}((\Delta^n)^{\text{op}}, \mathbf{Adm})$ .

Denote by  $\mathbf{Op}^{\wr}$  the following subcategory of  $E(\mathbf{Adm})$ . The objects of  $\mathbf{Op}^{\wr}$  are those pairs  $(m, X) \in E(\mathbf{Adm})$  such that for any integer  $1 \leq i \leq m$ , the nerve of the admissible functor  $X_i \rightarrow X_{i-1}$  is a coronal fibration, and the operator category  $X_0$  is equivalent to  $\{*\}$ . The morphisms  $(n, Y) \rightarrow (m, X)$  are those pairs  $(\phi, g)$  such that for any integer  $0 \leq i \leq n$ , the admissible functor  $Y_i \rightarrow X_{\phi(i)}$  is an operator morphism.

**3.9. Proposition.** *The inner fibration  $(\mathbf{Op}^{\wr})^{\text{op}} \rightarrow N\Delta^{\text{op}}$  is a monoidal  $\infty$ -category.*

*Proof.* We first show that  $\mathbf{Op}^{\wr} \rightarrow N\Delta$  is a cartesian fibration. Indeed, for any object  $(m, X)$  of  $\mathbf{Op}^{\wr}$  and any edge  $\phi: \mathbf{n} \rightarrow \mathbf{m}$  of  $N\Delta$ , there exists a morphism  $Z \rightarrow \phi^*X$  in which  $Z: (\Delta^n)^{\text{op}} \rightarrow \mathbf{Adm}$  is a diagram such that  $Z_0$  is equivalent to  $\{1\}$  and, for any integers  $0 \leq i \leq j \leq n$ , the square

$$\begin{array}{ccc} Z_j & \longrightarrow & X_{\phi(j)} \\ \downarrow & & \downarrow \\ Z_i & \longrightarrow & X_{\phi(i)} \end{array}$$

is a pullback square. It is straightforward now to check that the resulting edge  $Z \rightarrow X$  is cartesian over  $\phi$ .

It now remains to show that for any integer  $m \geq 0$ , the natural map

$$\sigma_m: \mathbf{Op}_m^{\wr} \rightarrow \prod_{i=1}^m \mathbf{Op}_{\{i-1, i\}}^{\wr}$$

is an equivalence of  $\infty$ -categories. This is the functor that assigns to any object  $(m, X)$  of  $\mathbf{Op}_m^{\wr}$  the tuple

$$(X(m)_1, X(m-1)_1, \dots, X(2)_1, X(1)),$$

where  $X(k)_1$  denotes the fiber of  $X(k) \rightarrow X(k-1)$  over a terminal object of  $X(k-1)$ . It follows from Lm. 3.3 that this functor is indeed an equivalence.  $\square$

**3.10. Example.** Of course we may form the wreath product of *any* two operator categories, but of particular import are the operator categories obtained by forming the iterated wreath product of  $\mathbf{O}$  with itself:

$$\mathbf{O}^{(n)} := \mathbf{O} \wr \mathbf{O} \wr \dots \wr \mathbf{O}.$$

As Clemens Berger has observed, the homotopy theory of weak  $\mathbf{O}^{(n)}$ -operads is a homotopy-coherent variant of Michael Batanin's notion of an  $n$ -operad [6]. We prove below that  $E_n$  is equivalent to the symmetrization  $\mathbf{Symm}(U_{\mathbf{O}^{(n)}})$  of the terminal weak  $\mathbf{O}^{(n)}$ -operad (§11). This is a variant of Batanin's result in [6].

## 4. BOARDMAN–VOGT TENSOR PRODUCTS AND WEAK ALGEBRAS

It is well-known that the classical Boardman–Vogt tensor product [8] exhibits better homotopical properties when it is extended to suitably weak operads, as in [16] and [13]. It can also be *externalized* over the wreath product construction given the previous section, as we now demonstrate.

**4.1. Notation.** For any operator categories  $\Phi$  and  $\Psi$ , consider the functor

$$\wr: \Psi \times \Phi \longrightarrow \Psi \wr \Phi$$

that carries the pair  $(J, I)$  to the object  $J \wr I := ((J_i)_{i \in |I|}; I)$ , in which  $J_i = J$  for each  $i \in |I|$ . This induces a functor

$$W: \Delta_\Psi \times_\Delta \Delta_\Phi \longrightarrow \Delta_{\Psi \wr \Phi}$$

given by the assignment

$$([J_0 \longrightarrow \cdots \longrightarrow J_m], [I_0 \longrightarrow \cdots \longrightarrow I_m]) \longmapsto [J_0 \wr I_0 \longrightarrow \cdots \longrightarrow J_m \wr I_m].$$

**4.2.** It is clear that for any operator morphisms  $H: \Psi' \longrightarrow \Psi$  and  $G: \Phi' \longrightarrow \Phi$ , the square

$$\begin{array}{ccc} \Delta_{\Psi'} \times_\Delta \Delta_{\Phi'} & \xrightarrow{W} & \Delta_{\Psi' \wr \Phi'} \\ \downarrow & & \downarrow \\ \Delta_\Psi \times_\Delta \Delta_\Phi & \xrightarrow{W} & \Delta_{\Psi \wr \Phi} \end{array}$$

commutes.

**4.3. Definition.** Suppose  $\Phi$  and  $\Psi$  operator categories, and suppose  $p: X \longrightarrow N\Delta_\Phi^{\text{op}}$  and  $q: Y \longrightarrow N\Delta_\Psi^{\text{op}}$  left fibrations. For any complete Segal  $\Psi \wr \Phi$ -operad  $Z$ , a **pairing**  $(X, Y) \longrightarrow Z$  is a commutative diagram

$$\begin{array}{ccc} Y \times_{N\Delta^{\text{op}}} X & \longrightarrow & Z \\ (q, p) \downarrow & & \downarrow \\ N\Delta_\Psi^{\text{op}} \times_{N\Delta^{\text{op}}} N\Delta_\Phi^{\text{op}} & \xrightarrow{W} & N\Delta_{\Psi \wr \Phi}^{\text{op}} \end{array}$$

Write  $\text{Pair}^{\Psi, \Phi}(Y, X; Z)$  for the simplicial set  $\text{Map}_{/N\Delta_{\Psi \wr \Phi}^{\text{op}}}(Y \times_{N\Delta^{\text{op}}} X, Z)$  of pairings  $(Y, X) \longrightarrow Z$ .

**4.4. Proposition.** *Suppose  $\Phi$  and  $\Psi$  operator categories, and suppose  $X \longrightarrow N\Delta_\Phi^{\text{op}}$  and  $Y \longrightarrow N\Delta_\Psi^{\text{op}}$  left fibrations. Then the functor*

$$\text{Pair}^{\Psi, \Phi}(Y, X; -): \mathbf{Operad}^{\Psi \wr \Phi, \text{op}} \longrightarrow \mathbf{Kan}$$

*is corepresentable.*

*Proof.* Denote by  $\mathbf{LFib}(S)$  the simplicial nerve of the simplicial category of left fibrations to a fixed simplicial set  $S$ . Denote by  $L: \mathbf{LFib}(N\Delta_{\Psi \wr \Phi}^{\text{op}}) \longrightarrow \mathbf{Operad}^{\Psi \wr \Phi}$  the left adjoint to the inclusion. Then  $\text{Pair}^{\Psi, \Phi}(Y, X; -)$  is corepresented by the object  $LW_!(Y \times_{N\Delta^{\text{op}}} X)$ .  $\square$

**4.5. Definition.** Suppose  $\Phi$  and  $\Psi$  operator categories, suppose  $X \longrightarrow N\Delta_\Phi^{\text{op}}$  and  $Y \longrightarrow N\Delta_\Psi^{\text{op}}$  left fibrations, and suppose  $Z$  a complete Segal  $\Psi \wr \Phi$ -operad. A pairing  $(Y, X) \longrightarrow Z$  will be said to **exhibit  $Z$  as the Boardman–Vogt tensor product**

of  $Y$  and  $X$  just in case, for every operad complete Segal  $\Psi \wr \Phi$ -operad  $Z'$  it induces an equivalence

$$\mathrm{Map}_{\mathbf{Operad}^{\Psi \wr \Phi}}(Z, Z') \xrightarrow{\sim} \mathrm{Pair}^{\Psi, \Phi}((Y, X), Z').$$

In the presence of such a pairing, we write  $Y^{\Psi} \otimes^{\Phi} X$  for  $Z$ . Using [12], one can organize this into a functor

$$-^{\Psi} \otimes^{\Phi} -: \mathbf{LFib}(N\Delta_{\Phi}^{\mathrm{op}})^{\mathrm{op}} \times \mathbf{LFib}(N\Delta_{\Psi}^{\mathrm{op}})^{\mathrm{op}} \longrightarrow \mathbf{Operad}^{\Psi \wr \Phi}.$$

**4.6. Warning.** In contrast with the Boardman–Vogt tensor product of ordinary symmetric operads, note that the order matters here: since the wreath product is noncommutative, the objects  $Y^{\Psi} \otimes^{\Phi} X$  and  $X^{\Phi} \otimes^{\Psi} Y$  are not even objects of the same  $\infty$ -category.

It follows immediately from 4.2 that the Boardman–Vogt tensor product is compatible with operator morphisms.

**4.7. Proposition.** *For any operator morphisms  $H: \Psi' \rightarrow \Psi$  and  $G: \Phi' \rightarrow \Phi$  and for any left fibrations  $X \rightarrow N\Delta_{\Phi'}^{\mathrm{op}}$  and  $Y \rightarrow N\Delta_{\Psi'}^{\mathrm{op}}$ , there is a canonical equivalence*

$$(H \wr G)_!(Y^{\Psi'} \otimes^{\Phi'} X) \simeq H_! Y^{\Psi} \otimes^{\Phi} G_! X$$

of complete Segal  $\Psi \wr \Phi$ -operads.

It is easy to see that the Boardman–Vogt tensor product preserves colimits separately in each variable. Consequently, we obtain the following.

**4.8. Proposition.** *Suppose  $\Phi$  and  $\Psi$  operator categories, and suppose  $X \rightarrow N\Delta_{\Phi}^{\mathrm{op}}$  and  $Y \rightarrow N\Delta_{\Psi}^{\mathrm{op}}$  left fibrations. Then the functors*

$$-^{\Psi} \otimes^{\Phi} X: \mathbf{LFib}(N\Delta_{\Psi}^{\mathrm{op}})^{\mathrm{op}} \longrightarrow \mathbf{Operad}^{\Psi \wr \Phi}$$

and

$$Y^{\Psi} \otimes^{\Phi} -: \mathbf{LFib}(N\Delta_{\Phi}^{\mathrm{op}})^{\mathrm{op}} \longrightarrow \mathbf{Operad}^{\Psi \wr \Phi}$$

each admit a right adjoint, denoted  $\mathbf{Alg}^{\Phi, \Psi \wr \Phi}(X, -)$  and  $\mathbf{Alg}^{\Psi, \Psi \wr \Phi}(Y, -)$ , respectively.

**4.9.** Hence for any two operator categories  $\Phi$  and  $\Psi$ , we obtain a pair of functors

$$\mathbf{Alg}^{\Psi, \Psi \wr \Phi}: \mathbf{LFib}(N\Delta_{\Psi}^{\mathrm{op}})^{\mathrm{op}} \times \mathbf{Operad}^{\Psi \wr \Phi} \longrightarrow \mathbf{LFib}(N\Delta_{\Phi}^{\mathrm{op}})$$

and

$$\mathbf{Alg}^{\Phi, \Psi \wr \Phi}: \mathbf{LFib}(N\Delta_{\Phi}^{\mathrm{op}})^{\mathrm{op}} \times \mathbf{Operad}^{\Psi \wr \Phi} \longrightarrow \mathbf{LFib}(N\Delta_{\Psi}^{\mathrm{op}}).$$

In fact, these functors are valued in the  $\infty$ -category of weak operads. That is, suppose  $\Phi$  and  $\Psi$  operator categories, suppose

$$X \rightarrow N\Delta_{\Phi}^{\mathrm{op}} \quad \text{and} \quad Y \rightarrow N\Delta_{\Psi}^{\mathrm{op}}$$

left fibrations, and suppose  $Z$  a complete Segal  $\Psi \wr \Phi$ -operad. Then the left fibration  $\mathbf{Alg}^{\Psi, \Psi \wr \Phi}(Y, Z) \rightarrow N\Delta_{\Phi}^{\mathrm{op}}$  is a complete Segal  $\Phi$ -operad, and the left fibration  $\mathbf{Alg}^{\Phi, \Psi \wr \Phi}(X, Z) \rightarrow N\Delta_{\Psi}^{\mathrm{op}}$  is a complete Segal  $\Psi$ -operad.

This is not an entirely formal matter. It suffices to prove this for  $X$  and  $Y$  corepresentable. In this case, one shows that forming the Boardman–Vogt tensor product of any corepresentable left fibration with any element of the set  $S_{\Phi}$  (or  $S_{\Psi}$ ) of Pr. 2.12 itself lies in the strongly saturated class generated by  $S_{\Psi \wr \Phi}$ . We leave these details to the reader.

Now let's concentrate on the situation in which one of the two complete Segal operads is the terminal operad.

**4.10. Notation.** Suppose  $\Phi$  and  $\Psi$  two operator categories. For any complete Segal  $\Psi \wr \Phi$ -operad  $Z$ , we will write  $\mathbf{Mon}^{\Phi, \Psi \wr \Phi}(Z)$  for  $\mathbf{Alg}^{\Phi, \Psi \wr \Phi}(U_\Phi, Z)$ , and we will write  $\mathbf{Mon}^{\Phi, \Psi \wr \Phi}(Z)$  for  $\mathbf{Alg}^{\Psi, \Psi \wr \Phi}(U_\Psi, Z)$ .

**4.11.** Suppose  $\Phi$  an operator category. Note that we have canonical equivalences  $\Phi \wr \{1\} \simeq \Phi \simeq \{1\} \wr \Phi$ , through which the functor  $\mathbf{Alg}^{\Phi, \Phi \wr \{1\}}$  and  $\mathbf{Alg}^{\Phi, \{1\} \wr \Phi}$  may be identified. We write

$$\mathbf{Alg}^\Phi : \mathbf{LFib}(N\Delta_\Phi^{\text{op}})^{\text{op}} \times \mathbf{Operad}^\Phi \longrightarrow \mathbf{LFib}(N\Delta^{\text{op}})$$

for the common functor. For any left fibration  $Y \longrightarrow N\Delta_\Phi^{\text{op}}$  and any complete Segal  $\Phi$ -operad  $Z$ , we will refer to objects of  $\mathbf{Alg}^\Phi(Y, Z)$  (i.e., those 0-simplices that lie over  $0 \in N\Delta^{\text{op}}$ ) as  *$Y$ -algebras in  $Z$* . If  $Y$  is the terminal  $\Phi$ -operad  $U_\Phi$ , then we will refer to  $Y$ -algebras in  $Z$  as *complete Segal  $\Phi$ -monoids in  $Z$* .

**4.12. Theorem.** *For any two operator categories  $\Phi$  and  $\Psi$ , the pairing*

$$(U_\Psi, U_\Phi) \longrightarrow U_{\Psi \wr \Phi}$$

*given by  $W$  exhibits  $U_{\Psi \wr \Phi}$  as the Boardman–Vogt tensor product of  $U_\Psi$  and  $U_\Phi$ .*

The proof is a bit involved, so it appears in an appendix; see §A.

**4.12.1. Corollary.** *For any two operator categories  $\Phi$  and  $\Psi$ , one has, for any  $\Psi \wr \Phi$ -operad  $Z$ , canonical equivalences*

$$\mathbf{Mon}^{\Psi \wr \Phi}(Z) \simeq \mathbf{Mon}^\Phi(\mathbf{Mon}^{\Psi, \Psi \wr \Phi}(Z)) \simeq \mathbf{Mon}^\Psi(\mathbf{Mon}^{\Phi, \Psi \wr \Phi}(Z)).$$

Roughly speaking, we have shown that complete Segal  $\Psi \wr \Phi$ -monoids are complete Segal  $\Psi$ -monoids in complete Segal  $\Phi$ -monoids, which in turn are complete Segal  $\Phi$ -monoids in complete Segal  $\Psi$ -monoids.

**4.13. Example.** The  $(\infty, 1)$ -category of complete Segal  $\mathbf{O}^{(n)}$ -monoids (in some complete Segal  $\mathbf{O}^{(n)}$ -operad  $Z$ ) is equivalent to the  $(\infty, 1)$ -category

$$\mathbf{Mon}^{\mathbf{O}, \mathbf{O}}(\mathbf{Mon}^{\mathbf{O}, \mathbf{O} \wr \mathbf{O}}(\dots \mathbf{Mon}^{\mathbf{O}, \mathbf{O}^{(n)}}(Z) \dots))$$

of complete Segal  $\mathbf{O}$ -monoids in complete Segal  $\mathbf{O}$ -monoids in  $\dots$  in complete Segal  $\mathbf{O}$ -monoids in  $Z$ . The assertion that the operad  $E_n$  is equivalent to the symmetrization of the terminal  $\mathbf{O}^{(n)}$ -operad (Pr. 11.4) thus states that if  $Z$  is a symmetric operad, then the homotopy theory of  $E_n$ -algebras in  $Z$  is equivalent to the homotopy theory of complete Segal  $\mathbf{O}$ -monoids in complete Segal  $\mathbf{O}$ -monoids in  $\dots$  in complete Segal  $\mathbf{O}$ -monoids in the underlying complete Segal  $\mathbf{O}^{(n)}$ -operad of  $Z$ .

**4.14. Example.** For any integers  $n \geq m \geq 0$ , we have an inclusion

$$s : \mathbf{O}^{(m)} \cong \{1\}^{(n-m)} \wr \mathbf{O}^{(m)} \hookrightarrow \mathbf{O}^{(n-m)} \wr \mathbf{O}^{(m)} \cong \mathbf{O}^{(n)},$$

which is a section of the coronal fibration  $p : \mathbf{O}^{(n)} \longrightarrow \mathbf{O}^{(m)}$ . We can thus form the colimit  $\mathbf{O}^{(\infty)} := \text{colim}_{n \geq 0} \mathbf{O}^{(n)}$ . This can be viewed as the category whose objects are sequences

$$(M_1, M_2, \dots)$$

with  $M_n \in \mathbf{O}^{(n)}$  such that for every for each  $n \geq 1$ , one has  $p(M_n) = M_{n-1}$  and for each  $n \gg 1$ , one has  $s(M_n) = M_{n+1}$ . Then  $\mathbf{O}^{(\infty)}$  is an operator category.

## 5. PERFECT OPERATOR CATEGORIES

The May–Thomason category of a symmetric operad [15] is a category that lives over Segal’s category  $\Gamma^{\text{op}}$  of pointed finite sets. Lurie’s theory of  $\infty$ -operads [13, Ch. 2] is built upon a generalization of this picture. How do we understand the relationship between the operator category  $\mathbf{F}$  and the category  $\Gamma^{\text{op}}$ ? For nonsymmetric operads, one has a similar May–Thomason construction over  $\Delta^{\text{op}}$  [22]. How do we understand the relationship between the operator category  $\mathbf{O}$  and the category  $\Delta^{\text{op}}$ ? Is it analogous to the relationship between the operator category  $\mathbf{F}$  and the category  $\Gamma^{\text{op}}$ ?

At first blush, the answer appears to be no:  $\Gamma^{\text{op}}$  is the category of pointed objects of  $\mathbf{F}$ , and  $\Delta^{\text{op}}$  is opposite of the category of nonempty objects of  $\mathbf{O}$ . Expressed this way,  $\Gamma^{\text{op}}$  and  $\Delta^{\text{op}}$  don’t look the least bit similar. However, an insight that we inherited from Tom Leinster [11, pp. 40–43] shows us how to think of them each as special cases of a general construction. This insight leads us to study a special class of operator categories, which we call *perfect*. We discuss the key properties of these operator categories here. In the next section we find that perfect operator categories admit a canonical monad, and in the section after that, we use that monad to define what we call the *Leinster category*  $\Lambda(\Phi)$  of a perfect operator category, and we show that

$$\Lambda(\mathbf{F}) \simeq \Gamma^{\text{op}} \quad \text{and} \quad \Lambda(\mathbf{O}) \simeq \Delta^{\text{op}}.$$

These Leinster categories will be the foundation upon which our analogue of Lurie’s theory of  $\infty$ -operads over operator categories is built.

The first property enjoyed by perfect operator categories is the existence of a *point classifier*. Point classifiers play a role in the theory of perfect operator categories that is in many respects analogous to the role played by the subobject classifier in topos theory.

**5.1. Definition.** Suppose  $B$  a pointed category, i.e., a category containing a zero object  $0$ . Then a morphism  $r: W \rightarrow V$  of  $B$  is *conservative* if the square

$$\begin{array}{ccc} 0 & \longrightarrow & W \\ \parallel & & \downarrow r \\ 0 & \longrightarrow & V \end{array}$$

is a pullback square. Clearly identity morphisms and composites of conservative morphisms are conservative, so this specifies a subcategory  $B^{\text{cons}}$  of  $B$ . A *point classifier* for  $B$  is a terminal object  $T$  of  $B^{\text{cons}}$ .

If a point classifier in a pointed category exists, then it is essentially unique.

**5.2. Proposition.** *Suppose  $B$  a pointed category containing all fibers over  $0$ . Suppose  $T$  a point classifier for  $B$ . Then there is a functor  $\chi: B^{\text{cons}} \rightarrow (B/T)$ , such that the square*

$$\begin{array}{ccc} B^{\text{cons}} & \xrightarrow{\chi} & (B/T) \\ \downarrow & & \downarrow \text{fib} \\ \{0\} & \longrightarrow & B \end{array}$$

is a pullback square, where  $\text{fib}: (B/T) \rightarrow B$  is the functor that assigns to any morphism  $V \rightarrow T$  its fiber.

*Proof.* Compose the equivalence of categories  $B^{\text{cons}} \xrightarrow{\sim} (B^{\text{cons}}/T)$  with the inclusion  $(B^{\text{cons}}/T) \hookrightarrow (B/T)$  to construct the functor  $\chi$ . Now the result follows from the universal property of  $T$ .  $\square$

In this work, point classifiers will be relevant only when the pointed category  $B$  in question is the category  $\Phi_*$  of pointed objects in an operator category  $\Phi$ .

**5.3. Definition.** Suppose  $\Phi$  an operator category. Suppose  $(T, t)$  a point classifier for  $\Phi_*$ . Then for any object  $(I, i)$  of  $\Phi_*$ , the unique conservative morphism  $(I, i) \rightarrow (T, t)$  will be called the *classifying morphism* for  $i$ , and will be denoted  $\chi_i$ . We shall call the point  $t \in |T|$  the *special point* of  $T$ , and for any morphism  $I \rightarrow T$  of  $\Phi$ , the fiber  $I_t$  will be called the *special fiber*. Write  $\text{fib}: (\Phi/T) \rightarrow \Phi$  for the *special fiber functor*  $I \mapsto I_t$ .

**5.4. Example.** (5.4.1) The category  $\{1\}$  trivially has a point classifier.

(5.4.2) The category  $\mathbf{O}_*$  of pointed ordered finite sets has a point classifier, namely,  $T_{\mathbf{O}} := \{0, 1, 2\}$ , wherein 1 is the marked point. Indeed, suppose  $(J, j)$  a pointed ordered finite set. Then there is a morphism  $\chi_j: J \rightarrow T_{\mathbf{O}}$  defined by the formula

$$\chi_j(k) := \begin{cases} 0 & \text{if } k < j; \\ 1 & \text{if } k = j; \\ 2 & \text{else.} \end{cases}$$

The fiber of  $\chi_j$  is of course the point  $j$  itself, and  $\chi_j$  is moreover unique with this property.

(5.4.3) The category  $\mathbf{F}_*$  of pointed finite sets has a point classifier, namely, the set  $T_{\mathbf{F}} := \{0, 1\}$ , wherein 1 is the marked point. Indeed, this is a consequence of the observation that  $T_{\mathbf{F}}$  is the subobject classifier for the topos of sets.

(5.4.4) The category  $(\mathbf{O} \wr \mathbf{O})_*$  has a point classifier. This is the object

$$(T_{\mathbf{O}}, \{*, T_{\mathbf{O}}, *\}),$$

which may be pictured thus:

$$T_{\mathbf{O}} : \begin{array}{c} * \quad T_{\mathbf{O}} \quad * \\ \bullet \\ \bullet \quad \circ \quad \bullet \\ \bullet \\ \hline \bullet \quad \bullet \quad \bullet \end{array}$$

This trend continues: the category  $(\mathbf{O}^{(n)})_*$  has a point classifier, which may be represented in  $\mathbf{R}^n$  as the special point at the origin and  $2n$  points at the intersection of the unit  $(n - 1)$ -sphere and the coordinate axes.

**5.5. Example.** Suppose  $F: \Psi \rightarrow \Phi$  a fully faithful operator morphism, and suppose  $(T, t)$  an object of  $\Psi_*$  such that  $F(T, t)$  is a point classifier for  $\Phi_*$ . Then  $(T, t)$  is a point classifier for  $\Psi_*$  as well.

It follows that for any integer  $m \geq 3$ , the category  $\mathbf{O}_{\leq m, *}$  has a point classifier, and for any integer  $n \geq 2$ , the category  $\mathbf{F}_{\leq n, *}$  has a point classifier.

**5.6. Definition.** An operator category  $\Phi$  is *perfect* if the following conditions are satisfied.

(5.6.1) The category  $\Phi_*$  contains a point classifier  $(T_\Phi, t_\Phi)$ .

(5.6.2) The special fiber functor  $\text{fib}: (\Phi/T_\Phi) \rightarrow \Phi$  admits a right adjoint  $E_\Phi$ .

One denotes the full subcategory of **Adm** (respectively, of **Op**) spanned by the perfect operator categories by  $\mathbf{Adm}^{\text{perf}}$  (resp.,  $\mathbf{Op}^{\text{perf}}$ ).

**5.7. Notation.** As a rule, we drop the subscripts from the notation for the point classifier and the right adjoint of  $\text{fib}$  if it is clear from the context which operator category is under consideration.

If  $\Phi$  is a perfect operator category, then let us abuse notation by writing  $T$  for the endofunctor of  $\Phi$  obtained by composing  $E: \Phi \rightarrow (\Phi/T_\Phi)$  with the forgetful functor  $(\Phi/T_\Phi) \rightarrow \Phi$ . Hence for some object  $I$  of  $\Phi$ , the notation  $EI$  will denote  $TI$  along with the *structure morphism*  $e_I: TI \rightarrow T$ . This abuse is partly justified by the following observation.

**5.8. Proposition.** *If  $\Phi$  is a perfect operator category, then the structural morphism  $e_*: T(*) \rightarrow T$  is an isomorphism.*

**5.8.1. Corollary.** *If  $\Phi$  is a perfect operator category, and  $I$  is an object of  $\Phi$ , then  $e_I = T(!)$ .*

**5.9. Example.** (5.9.1) Of course the operator category  $\{*\}$  is perfect.

(5.9.2) The operator category  $\mathbf{O}$  is perfect; the functor  $E$  assigns to any ordered finite set  $I$  the ordered finite set  $TI$  obtained by adding a single point at the beginning and a single point at the end, along with the unique map  $e_I: TI \rightarrow T$  whose special fiber is precisely  $I \subset TI$ .

(5.9.3) The operator category  $\mathbf{F}$  is perfect; the functor  $E$  assigns to any finite set  $I$  the finite set  $TI$  obtained by adding a disjoint basepoint to  $I$ , along with the unique map  $e_I: TI \rightarrow T$  whose special fiber is precisely  $I \subset TI$ .

(5.9.4) For any integer  $n \geq 1$ , neither  $\mathbf{O}_{\leq n}$  nor  $\mathbf{F}_{\leq n}$  are perfect.

(5.9.5) The operator category  $\mathbf{O} \wr \mathbf{O}$  is perfect. The functor  $T_{\mathbf{O} \wr \mathbf{O}}$  carries an object  $(I, M)$  to an object

$$(T_{\mathbf{O}}I, \{S_\eta\}_{\eta \in |T_{\mathbf{O}}I|}),$$

where  $S_\eta = 1$  if  $\eta$  is either of the endpoints in  $T_{\mathbf{O}}I$  and  $S_\eta = T_{\mathbf{O}}M_i$  if  $\eta = i \in |I| \subset |T_{\mathbf{O}}I|$ .

More generally, wreath products of perfect operator categories are perfect.

**5.10. Proposition.** *Suppose  $\Phi$  and  $\Psi$  two perfect operator categories. Then the operator category  $\Psi \wr \Phi$  is perfect as well.*

*Proof.* Choose point classifiers  $(T_\Phi, t_\Phi)$  of  $\Phi$  and  $(T_\Psi, t_\Psi)$  of  $\Psi$  and a terminal object  $1 \in \Psi$ . Consider an object  $T_{\Psi \wr \Phi} = (T_\Phi, \{S_\eta\}_{\eta \in |T_\Phi|})$  of  $\Psi \wr \Phi$  in which

$$S_\eta = \begin{cases} 1 & \text{if } \eta \neq t_\Phi; \\ T_\Psi & \text{if } \eta = t_\Phi. \end{cases}$$

Consider the point  $t_{\Psi \wr \Phi} := (t_\Phi, t_\Psi) \in |T_{\Psi \wr \Phi}|$ . Clearly the pair  $(T_{\Psi \wr \Phi}, t_{\Psi \wr \Phi})$  is a point classifier of  $\Psi \wr \Phi$ .

The right adjoint  $E_{\Psi \wr \Phi}$  of the special fiber functor  $\text{fib}$  is defined by carrying any object  $(I, \{M_i\}_{i \in |I|})$  to the object  $(T_\Phi I, \{N_j\}_{j \in |T_\Phi I|})$ , where

$$N_j = \begin{cases} 1 & \text{if } j \notin |I| \subset |T_\Phi I|; \\ T_\Psi M_j & \text{if } j \in |I| \subset |T_\Phi I|, \end{cases}$$

along with the morphism

$$(T_\Phi I, \{N_j\}_{j \in |T_\Phi I|}) \longrightarrow (T_\Phi, \{S_\eta\}_{\eta \in |T_\Phi|})$$

given by  $e_I: T_\Phi I \longrightarrow T_\Phi$  and, for any  $j \in |T_\Phi I|$ , the morphism  $e_{M_j}: T_\Psi M_j \longrightarrow T_\Psi$  when  $j \in |I|$ , and the identity map on 1 when  $j \notin |I|$ .  $\square$

This allows us to restrict the monoidal structure of Pr. 3.9 to a monoidal structure on  $\mathbf{Op}^{\text{perf}}$ .

**5.10.1. Corollary.** *Denote by  $\mathbf{Op}^{\text{perf}, \lambda}$  the full subcategory of  $\mathbf{Op}^\lambda$  spanned by those pairs  $(m, X)$  such that each  $X_i$  is perfect. Then the composite  $(\mathbf{Op}^{\text{perf}, \lambda})^{\text{op}} \longrightarrow N\Delta^{\text{op}}$  is a monoidal  $\infty$ -category.*

Consequently, we obtain the following example.

**5.11. Example.** The operator categories  $\mathbf{O}^{(n)}$  are all perfect.

## 6. THE CANONICAL MONAD ON A PERFECT OPERATOR CATEGORY

A perfect operator category  $\Phi$  always comes equipped with a monad. In effect, this monad adds points “in every direction” in  $\Phi$ ; these “directions” are indexed by the “non-special” points of the point classifier.

**6.1.** Observe that for any object  $I$  of an operator category  $\Phi$ , and any point  $i \in |I|$ , the fiber functor  $(-)_i$  is already right adjoint to the fully faithful functor  $p_i: \Phi \longrightarrow (\Phi/I)$ , which sends any object  $J$  of  $\Phi$  to the composite

$$J \longrightarrow \{i\} \hookrightarrow I.$$

Observe that  $\Phi$  thus has the structure of a localization of  $(\Phi/I)$ ; that is, the unit  $J \longrightarrow p_i(J)_i$  is an isomorphism.

If  $\Phi$  is perfect, there is a string of adjoints

$$\Phi \begin{array}{c} \xrightarrow{p_t} \\ \xleftrightarrow{\quad} (\Phi/T) \\ \xleftarrow{E} \end{array}$$

and as the following result shows,  $\Phi$  is both a localization and a colocalization of  $(\Phi/T)$ .

**6.2. Lemma.** *If  $\Phi$  is a perfect operator  $\mathbf{X}$ -category, then the adjoint pair  $(\text{fib}, E)$  gives  $\Phi$  the structure of a colocalization of  $(\Phi/T)$ .*

*Proof.* The claim is simply that the counit  $\kappa: \text{fib} \circ E \longrightarrow \text{id}_\Phi$  is an isomorphism. The inverse to the unit  $\text{id}_\Phi \longrightarrow \text{fib} \circ p_t$  induces a morphism  $\epsilon: p_t \longrightarrow E$  of functors  $\Phi \longrightarrow (\Phi/T)$ ; this gives a morphism  $\iota: \text{id}_\Phi \longrightarrow T$  of endofunctors of  $\Phi$ .

Now for any object  $I$  of  $\Phi$ , the resulting square

$$\begin{array}{ccc} I & \longrightarrow & TI \\ \downarrow & & \downarrow \\ \{t\} & \hookrightarrow & T \end{array}$$

is a pullback square; indeed, a commutative diagram

$$\begin{array}{ccc} I' & \longrightarrow & TI \\ \downarrow & & \downarrow \\ \{t\} & \hookrightarrow & T \end{array}$$

is the same data as a morphism  $p_t I' \rightarrow EI$  of  $(\Phi/T)$ , which is in turn the same data as a  $I' \cong (p_t I')_t \rightarrow I$ .  $\square$

If  $\Phi$  is a perfect operator category, then, in effect, the endofunctor  $T: \Phi \rightarrow \Phi$ , when applied to an object of  $\Phi$ , adds as few points as possible in as many directions as possible. It turns out that this endofunctor is a monad; let us now construct a natural transformation  $\mu: T^2 \rightarrow T$  that, together with the natural transformation  $\iota: \text{id}_\Phi \rightarrow T$  from the previous proof, will exhibit a monad structure on  $T$ .

**6.3. Notation.** Suppose  $\Phi$  a perfect operator category. The embedding  $\iota_T: T \hookrightarrow TT$  permits us to regard the special point  $t \in |T|$  as a point of  $TT$  as well. Now consider the classifying morphism  $\chi_t: TT \rightarrow T$ . Its special fiber is the point  $\iota_T(t)$ , so that the following diagram is a pullback:

$$\begin{array}{ccc} \{t\} & \hookrightarrow & T \xrightarrow{\iota_T} TT \\ \parallel & & \downarrow \chi_t \\ \{t\} & \hookrightarrow & T \end{array}$$

**6.4. Notation.** For any object  $I$  of a perfect operator category  $\Phi$ , the functor  $E$  induces a functor

$$E_{/I}: (\Phi/I) \rightarrow (\Phi/TI),$$

which is right adjoint to the functor  $\iota_I^{-1} := - \times_{TI} I: (\Phi/TI) \rightarrow (\Phi/I)$ .

**6.5. Lemma.** *If  $\Phi$  is a perfect operator category, then there is an isomorphism of functors*

$$\rho: \text{fib} \xrightarrow{\simeq} \text{fib} \circ \chi_{t,!} \circ E_{/TT},$$

where  $\chi_{t,!}: (\Phi/TT) \rightarrow (\Phi/T)$  is the functor given by composition with  $\chi_t$ .

*Proof.* For any morphism  $\phi: J \rightarrow T$ , every square of the diagram

$$\begin{array}{ccccc} J_t & \hookrightarrow & J & \hookrightarrow & TJ \\ \downarrow & & \downarrow \phi & & \downarrow T\phi \\ \{t\} & \hookrightarrow & T & \xrightarrow{\iota_T} & TT \\ \parallel & & & & \downarrow \chi_t \\ \{t\} & \hookrightarrow & & & T \end{array}$$

is a pullback square.  $\square$

**6.6.** By adjunction, we obtain a natural transformation  $\sigma: \chi_{t,!} \circ E_{/T} \rightarrow E \circ \text{fib}$ . If  $\phi: I \rightarrow T$  is a morphism of  $\Phi$ , then one can apply  $T$  to this morphism to obtain a morphism  $T(\phi): TI \rightarrow TT$ . One can, alternatively apply  $T$  to the special fiber  $I_t$

to obtain a morphism  $T(I_t) \rightarrow T$ . The component  $\sigma_I$  then fits into a commutative square

$$\begin{array}{ccc} TI & \xrightarrow{\sigma_I} & T(I_t) \\ T(\phi) \downarrow & & \downarrow e_{I_t} \\ TT & \xrightarrow{\chi_t} & T \end{array}$$

Here, the special fiber of  $I$  inside  $TI$  is mapped isomorphically to the special fiber of  $T(I_t)$  under  $\sigma_I$ .

**6.7. Definition.** Suppose  $\Phi$  a perfect operator category. Define a morphism of endofunctors  $\mu: T^2 \rightarrow T$  as the composition

$$T^2 = U \circ E \circ U \circ E = U \circ \chi_{t,!} \circ E_{/T} \circ E \xrightarrow{\text{id} \circ \sigma \circ \text{id}} U \circ E \circ \text{fib} \circ E \xrightarrow{\text{id} \circ \kappa} U \circ E = T,$$

where  $\kappa: \text{fib} \circ E \rightarrow \text{id}_\Phi$  is the counit isomorphism (Lm. 6.2).

**6.8.** More explicitly, if  $I$  is an object of  $\Phi$ , then one has, following 6.6, a commutative diagram

$$(6.8.1) \quad \begin{array}{ccc} T^2 I & \xrightarrow{\sigma_{TI}} & T((TI)_t) \xrightarrow{\cong} TI \\ T(e_I) \downarrow & & \downarrow \\ TT & \xrightarrow{\chi_t} & T \end{array}$$

The composite  $T^2 I \rightarrow TI$  is the component  $\mu_I$ .

**6.9. Theorem.** *The endofunctor  $T$  on a perfect operator category  $\Phi$  is a monad with unit  $\iota: \text{id}_\Phi \rightarrow T$  and multiplication  $\mu: T^2 \rightarrow T$ .*

The proof, though quite elementary, is a little involved, so we postpone it (§B).

For now, let us turn to the functoriality of this monad structure in admissible functors.

**6.10.** An admissible functor  $F: \Psi \rightarrow \Phi$  between perfect operator categories induces a functor

$$F_{/T_\Psi}: (\Psi/T_\Psi) \rightarrow (\Phi/FT_\Psi),$$

and there is a unique conservative morphism  $\chi_{F(t_\Psi)}: (FT_\Psi, F(t_\Psi)) \rightarrow (T_\Phi, t_\Phi)$  of  $\Phi_*$ . So let  $F_{/T}$  (with no subscript on  $T$ ) denote the composite

$$(\Psi/T_\Psi) \xrightarrow{F_{/T_\Psi}} (\Phi/FT_\Psi) \xrightarrow{\chi_{F(t_\Psi),!}} (\Phi/T_\Phi).$$

We now have the following trivial observations.

**6.11. Lemma.** *For any admissible functor  $F: \Psi \rightarrow \Phi$  between perfect operator categories, there is a natural isomorphism  $\beta_F: \text{fib}_\Phi \circ F_{/T} \xrightarrow{\sim} F \circ \text{fib}_\Psi$ .*

**6.12. Lemma.** *For any admissible functor  $F: \Psi \rightarrow \Phi$  between perfect operator categories, the natural transformation  $F \rightarrow \text{fib}_\Phi \circ F_{/T} \circ E_\Psi$  corresponding to the isomorphism  $\beta_F$  by adjunction is itself an isomorphism.*

Adjoint to the inverse of this isomorphism is a natural transformation

$$\alpha_F: F_{/T} \circ E_\Psi \rightarrow E_\Phi \circ F.$$

It may be characterized as follows.

6.13. **Proposition.** *For any object  $I$  of  $\Psi$ , there is a unique commutative square*

$$\begin{array}{ccc} FT_{\Psi}I & \xrightarrow{\alpha_{F,I}} & T_{\Phi}FI \\ \downarrow & & \downarrow \\ FT_{\Psi} & \xrightarrow{\chi_{F(t_{\Psi})}} & T_{\Phi} \end{array}$$

of  $\Phi$  whose special fiber is the square

$$\begin{array}{ccc} FI & \xlongequal{\quad} & FI \\ \downarrow & & \downarrow \\ F\{t_{\Psi}\} & \xlongequal{\quad} & \{t_{\Phi}\} \end{array}$$

The natural transformation  $\alpha_F$  is generally not an isomorphism, but it does behave well with respect to the monad structure.

6.14. More precisely, recall that if  $C$  and  $D$  are categories equipped with monads  $(T_C, \iota_C, \mu_C)$  and  $T_D$ , then a *colax morphism of monads*  $(F, \eta): (C, T_C) \rightarrow (D, T_D)$  is a functor  $F: C \rightarrow D$  equipped with a natural transformation  $\eta: FT_C \rightarrow T_DF$  with the property that the diagrams

$$(6.14.1) \quad \begin{array}{ccc} & \eta_{T_C} \swarrow & FT_C^2 \\ & & \searrow F\mu_C \\ T_DF T_C & & FT_C \\ T_C \eta \downarrow & & \swarrow \eta \\ T_D^2 F & & T_DF \\ & \mu_D F \searrow & \end{array} \quad \text{and} \quad \begin{array}{ccc} & & FT_C \\ & F \swarrow & \downarrow \eta \\ F & & T_DF \\ & \iota_F \searrow & \end{array}$$

of  $\text{Fun}(C, D)$  commute.

Thus the interaction of the natural transformation  $\alpha_F$  with the monad structures on  $\Psi$  and  $\Phi$  is summarized by the following result.

6.15. **Theorem.** *An admissible functor  $F: \Psi \rightarrow \Phi$  of perfect operator categories induces a colax morphism of monads  $(F, \alpha_F): (\Psi, T_{\Psi}) \rightarrow (\Phi, T_{\Phi})$ .*

Again we postpone the proof (§C).

6.16. Observe that the uniqueness of conservative morphisms with target  $(T_{\Phi}, t_{\Phi})$  implies that if  $G: X \rightarrow \Psi$  is another admissible functor of perfect operator  $\mathbf{X}$ -categories, then  $(F \circ G)_{/T} = F_{/T} \circ G_{/T}$ . Hence the assignment  $\Phi \mapsto (\Phi/T)$  defines a functor  $\mathbf{Adm}^{\text{perf}} \rightarrow \mathbf{Cat}$ . On the other hand, there is a forgetful functor  $\mathbf{Adm}^{\text{perf}} \rightarrow \mathbf{Cat}$ .

Now the functor  $\beta$  of Lm. 6.11 can be regarded as a natural transformation from the functor  $\Phi \mapsto (\Phi/T)$  to the forgetful functor  $\mathbf{Adm}^{\text{perf}} \rightarrow \mathbf{Cat}$  such that for any perfect operator category  $\Phi$ , one has  $\beta_{\Phi} = \text{fib}_{\Phi}$ . Similarly,  $\alpha$  can be regarded as a lax natural transformation from the forgetful functor  $\mathbf{Adm}^{\text{perf}} \rightarrow \mathbf{Cat}$  to the functor  $\Phi \mapsto (\Phi/T)$  such that for any perfect operator category  $\Phi$ , one has  $\alpha_{\Phi} = E_{\Phi}$ .

6.17. A 2-morphism  $\xi: (F, \eta) \longrightarrow (F', \eta')$  of colax morphisms of monads is an isomorphism of functors  $\xi: F \longrightarrow F'$  such that the square

$$\begin{array}{ccc} FT_C & \xrightarrow{\eta} & T_D F \\ \xi T_C \downarrow & & \downarrow T_D \xi \\ F' T_C & \xrightarrow{\eta} & T_D F' \end{array}$$

commutes.

Composition and identities are defined in the obvious manner; hence this defines a 2-category  $\mathbf{Mnd}_{\text{colax}}$  of small categories with monads and colax functors. By applying the nerve of each Mor-groupoid and taking the simplicial nerve of the resulting simplicial category, we obtain a quasicategory  $\mathbf{Mnd}_{\text{colax}}$ , which is in fact a 2-category in the sense of [12, §2.3.4].

We may summarize Th. 6.9 and Th. 6.15 together by stating that the assignment  $\Phi \longmapsto (\Phi, T)$  defines a functor  $\mathbf{Adm}^{\text{perf}} \longrightarrow \mathbf{Mnd}_{\text{colax}}$ .

## 7. LEINSTER CATEGORIES

The Kleisli category of the canonical monad on a perfect operator category  $\Phi$  is the category of free algebras for this monad. It can be thought of as indexing both operations in  $\Phi$  as well as projection maps in a coherent manner. In the examples of interest, this Kleisli category recovers a number of combinatorial categories familiar to homotopy theorists.

In particular, we'll find that the Kleisli category of the canonical monad on  $\mathbf{F}$  is Segal's  $\Gamma^{\text{op}}$ , the Kleisli category of the canonical monad on  $\mathbf{O}$  is  $\Delta^{\text{op}}$ , and the Kleisli category of the canonical monad on  $\mathbf{O}^{(n)}$  is Joyal's  $\Theta_n^{\text{op}}$ .

**7.1. Definition.** Suppose  $\Phi$  a perfect operator category. Then the *Leinster category*  $\Lambda(\Phi)$  of  $\Phi$  is the Kleisli category of the monad  $T_\Phi$ .

**7.2.** That is, the objects of the Leinster category of a perfect operator category  $\Phi$  are precisely those of  $\Phi$  itself, and for any two objects  $I$  and  $J$ ,

$$\text{Mor}_{\Lambda(\Phi)}(J, I) := \text{Mor}_\Phi(J, TI).$$

The identity at an object  $I$  is the unit  $\iota_I: I \longrightarrow TI$ . The composition law is defined by the composite

$$\begin{array}{ccc} \text{Mor}_\Phi(K, TJ) \times \text{Mor}_\Phi(J, TI) & \xrightarrow{\text{id} \times T_\Phi} & \text{Mor}_\Phi(K, TJ) \times \text{Mor}_\Phi(TJ, T^2 I) \\ & \xrightarrow{\circ} & \text{Mor}_\Phi(K, T^2 I) \\ & \xrightarrow{\mu_{I,*}} & \text{Mor}_\Phi(K, TI) \end{array}$$

for objects  $I, J$ , and  $K$ .

**7.3.** Suppose now  $F: \Psi \longrightarrow \Phi$  an admissible functor of perfect operator categories. Since colax morphisms of monoids induce functors of their Kleisli categories — and in fact this is a functor  $\mathbf{Mnd}_{\text{colax}} \longrightarrow \mathbf{Cat}$  —, it follows that  $f$  induces a functor  $\Lambda(F): \Lambda(\Psi) \longrightarrow \Lambda(\Phi)$  of the Leinster categories.

The admissible functor  $F$  induces a functor  $\Lambda(F): \Lambda(\Psi) \rightarrow \Lambda(\Phi)$  such that the map on objects is simply

$$\mathrm{Obj} \Lambda(F) = \mathrm{Obj} F: \mathrm{Obj} \Lambda(\Psi) = \mathrm{Obj} \Psi \rightarrow \mathrm{Obj} \Phi = \mathrm{Obj} \Lambda(\Phi),$$

and the map on morphisms is given by

$$\mathrm{Mor}_{\Psi}(J, T_{\Psi}I) \rightarrow \mathrm{Mor}_{\Phi}(FJ, FT_{\Psi}I) \xrightarrow{\alpha_{F,*}} \mathrm{Mor}_{\Phi}(FJ, T_{\Phi}FI)$$

(Pr. 6.13).

Consequently, the Leinster category construction defines a functor

$$\Lambda: \mathbf{Adm}^{\mathrm{perf}} \rightarrow \mathbf{Cat}.$$

**7.4. Example.** For any object  $I$  of a perfect operator category  $\Phi$  and for any point  $i \in |I|$ , the classifying morphism  $\chi_i: I \rightarrow T$  of  $\Phi$  is a morphism  $I \rightarrow \{i\}$  of  $\Lambda(\Phi)$ . In particular, of course,  $\{i\}$  is not terminal in  $\Lambda(\Phi)$ .

**7.5. Example.** The Leinster category of  $\mathbf{F}$  is Segal's category  $\Gamma^{\mathrm{op}}$  of pointed finite sets [1, 19].

**7.6. Example.** Let us study the Leinster category of  $\mathbf{O}$ . Denote by  $\perp$  and  $\top$  the two points of  $T(\emptyset)$ , so that

$$T(\emptyset) = \{\perp, \top\};$$

these correspond to two wide morphisms  $T \rightarrow T(\emptyset)$ , and for any object  $I$  of  $\Lambda(\mathbf{O})$ , they give rise to a map

$$c_I := (\perp!, \top!): \mathrm{Mor}_{\Lambda(\mathbf{O})}(I, *) \rightarrow \mathrm{Mor}_{\Lambda(\mathbf{O})}(I, \emptyset) \times \mathrm{Mor}_{\Lambda(\mathbf{O})}(I, \emptyset).$$

The map  $c_I$  is injective, so for any  $\phi, \psi \in \mathrm{Mor}_{\Lambda(\mathbf{O})}(I, \emptyset)$ , denote by  $\phi \star \psi$  the unique element of  $\mathrm{Mor}_{\Lambda(\mathbf{O})}(I, *)$  such that  $c_I(\phi \star \psi) = (\phi, \psi)$ , if it exists.

Then the subset  $c_I(\mathrm{Mor}_{\Lambda(\mathbf{O})}(I, *)) \subset \mathrm{Mor}_{\Lambda(\mathbf{O})}(I, \emptyset) \times \mathrm{Mor}_{\Lambda(\mathbf{O})}(I, \emptyset)$  is a total ordering  $\leq$  on  $\mathrm{Mor}_{\Lambda(\mathbf{O})}(I, \emptyset)$ ; in particular, for any  $\phi, \psi \in \mathrm{Mor}_{\Lambda(\mathbf{O})}(I, \emptyset)$ , the element  $\psi$  is the successor of  $\phi$  if and only if  $\phi \star \psi = \chi_i$  for some point  $i \in |I|$ .

The functor  $\mathrm{Mor}_{\Lambda(\mathbf{O})}(-, \emptyset)$  thus defines a functor  $\Lambda(\mathbf{O})^{\mathrm{op}} \rightarrow \Delta$ , where  $\Delta$  is defined as the full subcategory of  $\mathbf{O}$  consisting of nonempty objects; a quasiinverse functor is given by

$$\mathbf{n} \mapsto \mathbf{n}^{\vee} := \mathrm{Mor}_{\Delta}(\mathbf{n}, \mathbf{1}).$$

We deduce that  $\Lambda(\mathbf{O})$  is equivalent to  $\Delta^{\mathrm{op}}$ . (This observation goes back at least to Street [21].)

The following result is trivial to prove.

**7.7. Proposition.** *Suppose  $p: \Phi' \rightarrow \Phi$  an admissible functor between perfect operator categories that is also a Grothendieck fibration that classifies a functor  $\Phi^{\mathrm{op}} \rightarrow \mathbf{Adm}^{\mathrm{perf}}$ . Then the functor  $\Lambda(\Phi') \rightarrow \Lambda(\Phi)$  is a Grothendieck fibration such that for any object  $I \in \Phi$ , one has  $(\Lambda(\Phi'))_I \simeq \Lambda(\Phi'_I)$ .*

**7.8. Example.** The previous result now immediately implies that the Leinster category of the iterated wreath product  $\mathbf{O}^{(n)}$  coincides with Berger's iterated wreath product  $(\Delta \wr \Delta \wr \cdots \wr \Delta)^{\mathrm{op}}$  [7]. In particular,  $\Lambda(\mathbf{O}^{(n)})$  is equivalent to Joyal's category  $\Theta_n^{\mathrm{op}}$ .

## 8. QUASIOPERADS AND THEIR ALGEBRAS

When  $\Phi$  is a perfect operator category, the theory of weak  $\Phi$ -operads introduced above can also be codified in a manner similar to Jacob Lurie's theory of  $\infty$ -operads [13, Ch. 2]. Here, we explain how to generalize the basic elements of Lurie's theory to any perfect operator category; in most cases the proofs are trivial extensions of the proofs of loc. cit.

The following terminology was first introduced on  $\Gamma^{\text{op}}$  by Jacob Lurie.

**8.1. Definition.** Suppose  $\Phi$  a perfect operator category. We call a morphism  $J \rightarrow I$  of  $\Lambda(\Phi)$  *inert* if, the corresponding morphism  $J \rightarrow TI$  in  $\Phi$  has the property that the natural morphism

$$J \times_{TI} I \rightarrow I$$

is an isomorphism. Denote by  $\Lambda^\dagger(\Phi)$  the collection of inert morphisms of  $\Lambda(\Phi)$ . We will simply write  $N\Lambda(\Phi)$  for the marked simplicial set  $(N\Lambda(\Phi), \Lambda^\dagger(\Phi))$ .

Let us call a morphism  $J \rightarrow I$  of  $\Lambda(\Phi)$  *active* if the corresponding morphism  $J \rightarrow TI$  of  $\Phi$  factors as

$$J \hookrightarrow TJ \rightarrow TI,$$

where the morphism  $TJ \rightarrow TI$  is of the form  $T\phi$  for some morphism  $\phi: J \rightarrow I$  of  $\Phi$ . Denote by  $\Lambda_\dagger(\Phi)$  the collection of active morphisms of  $\Lambda(\Phi)$ .

**8.2.** Suppose  $\Phi$  a perfect operator category. We observe the following.

(8.2.1) A morphism  $J \rightarrow I$  of  $\Lambda(\Phi)$  is active just in case the corresponding morphism  $J \rightarrow TI$  in  $\Phi$  has the property that the natural morphism

$$J \times_{TI} I \rightarrow J$$

is an isomorphism.

(8.2.2) A morphism of  $\Lambda(\Phi)$  is both inert and active just in case it is an isomorphism.

Suppose  $\psi: K \rightarrow J$  and  $\phi: J \rightarrow I$  morphisms of  $\Lambda(\Phi)$ .

(8.2.3) If  $\psi$  is inert, then  $\phi \circ \psi$  is inert just in case  $\phi$  is.

(8.2.4) Dually, if  $\phi$  is active, then  $\phi \circ \psi$  is active just in case  $\psi$  is.

It is a result of Christopher Schommer-Pries that  $\Lambda(\Phi)$  always admits an inert-active factorization system. Clemens Berger has introduced the concept of a *moment category* that codifies the salient features of this factorization system. We intend to explore these ideas elsewhere. For now, we simply record the result of Schommer-Pries.

**8.3. Lemma** (C. Schommer-Pries). *Every morphism  $J \rightarrow I$  of the Leinster category of a perfect operator category  $\Phi$  admits factorization  $J \rightarrow K \rightarrow I$  into an inert morphism  $J \rightarrow K$  followed by an active morphism  $K \rightarrow I$ . Moreover, this factorization is unique up to unique isomorphism*

*Proof.* Using the structural morphism  $e_I: TI \rightarrow I$ , regard the morphism  $J \rightarrow TI$  as a morphism over  $T$ . Set  $K := J \times_{TI} I$ ; the projection  $K \rightarrow I$  in  $\Phi$  induces an active morphism  $K \rightarrow I$  in  $\Lambda(\Phi)$ . Now the universal property of  $T$  states that the set of maps  $J \rightarrow TK$  over  $T$  is in bijection with the set of maps  $J_t \rightarrow K$ . Hence we may choose the canonical isomorphism  $J_t \cong K$ , yielding a morphism  $J \rightarrow TK$ . It is obvious from the construction that this now gives the desired

factorization  $J \rightarrow K \rightarrow I$  of the original morphism, and the uniqueness follows from the observations above.  $\square$

**8.4. Example.** When  $\Phi = \mathbf{F}$ , a pointed map  $\phi: J_+ \rightarrow I_+$  of  $\Gamma^{\text{op}}$  corresponds to an inert morphism in our sense just in case it is inert in the sense of Lurie, that is, just in case the inverse image  $\phi^{-1}(i)$  of any point  $i \in I$  is a singleton. It corresponds to an active morphism in our sense just in case it is active in the sense of Lurie, that is, just in case the inverse image of the base point is a singleton.

**8.5. Example.** When  $\Phi = \mathbf{O}$ , a morphism  $\phi: \mathbf{m} \rightarrow \mathbf{n}$  of  $\Delta^{\text{op}}$  corresponds to an inert morphism in our sense just in case it corresponds to an injection  $\mathbf{n} \rightarrow \mathbf{m}$  given by the formula  $i \mapsto i + k$  for some fixed integer  $k \geq 1$ . It corresponds to an active morphism in our sense just in case it corresponds to a map  $\mathbf{n} \rightarrow \mathbf{m}$  that carries 0 to 0 and  $n$  to  $m$ .

**8.6. Proposition.** *The functor  $\Lambda(\Psi) \rightarrow \Lambda(\Phi)$  induced by an admissible functor  $F: \Psi \rightarrow \Phi$  of perfect operator categories preserves inert morphisms.*

*Proof.* Suppose  $J \rightarrow I$  an inert morphism of  $\Lambda(\Phi)$ . Consider the rectangle

$$\begin{array}{ccccc} F(J \times_{TI} I) & \xrightarrow{\cong} & FI & \xlongequal{\quad} & FI \\ \downarrow & & \downarrow & & \downarrow \\ FJ & \longrightarrow & FTI & \xrightarrow{\alpha_{F,I}} & TFI. \end{array}$$

The left hand square is a pullback since  $F$  is admissible; the morphism

$$F(J \times_{TI} I) \rightarrow FI$$

is an isomorphism because  $J \rightarrow I$  is inert; and the fact that right hand square is a pullback follows from the characterization of  $\alpha_{F,I}$  given in Pr. 6.13.  $\square$

**8.7. Notation.** Suppose  $X$  is an  $\infty$ -category, suppose  $S$  a 1-category, and suppose  $q: X \rightarrow S$  a functor. Suppose  $x, y \in X$ , and suppose  $g: q(x) \rightarrow q(y)$  a morphism of  $S$ . Denote by  $\text{Map}_X^g(x, y)$  the union of the connected components of  $\text{Map}_X(x, y)$  lying over the connected component of  $g$  in  $\text{Map}_S(q(x), q(y))$ .

We can now define the notion of  $\Phi$ -quasioperad in exact analogy with Lurie [13, Df. 2.1.1.10].

**8.8. Definition.** Suppose  $\Phi$  an operator category. Then a  $\Phi$ -*quasioperad* or  $\infty$ -*operad over*  $\Phi$  is an inner fibration  $p: X^\otimes \rightarrow N\Lambda(\Phi)$  satisfying the following conditions.

(8.8.1) For every morphism  $\phi: J \rightarrow I$  of  $\Lambda^\dagger(\Phi)$  and every object  $x \in X_J^\otimes$ , there is a  $p$ -cocartesian edge  $x \rightarrow y$  in  $X^\otimes$  covering  $\phi$ .

(8.8.2) For any objects  $I, J \in \Phi$ , any objects  $x \in X_I^\otimes$  and  $y \in X_J^\otimes$ , any morphism  $\phi: J \rightarrow I$  of  $\Lambda(\Phi)$ , and any  $p$ -cocartesian edges  $\{y \rightarrow y_i \mid i \in |I|\}$  lying over the inert morphisms  $\{\rho_i: I \rightarrow \{i\} \mid i \in |I|\}$ , the induced map

$$\text{Map}_{X^\otimes}^\phi(x, y) \rightarrow \prod_{i \in |I|} \text{Map}_{X^\otimes}^{\rho_i \circ \phi}(x, y_i)$$

is an equivalence.

(8.8.3) For any object  $I \in \Phi$ , the  $p$ -cocartesian morphisms lying over the inert morphisms  $\{I \rightarrow \{i\} \mid i \in |I|\}$  together induce an equivalence

$$X_I^\otimes \rightarrow \prod_{i \in |I|} X_{\{i\}}^\otimes.$$

8.9. **Example.** When  $\Phi = \mathbf{F}$ , the above definition coincides with Lurie's definition of  $\infty$ -operad [13].

8.10. **Example.** When  $\Phi = \{1\}$ , the conditions of the above definition are trivial, and we are simply left with the notion of a quasicategory.

In light of [13, Pr. 2.1.2.5], the inert-active factorization on the Leinster category of a perfect operator category lifts to any  $\infty$ -operad over it.

8.11. **Definition.** Suppose  $\Phi$  a perfect operator category, and suppose

$$p: X^\otimes \rightarrow N\Lambda(\Phi)$$

a  $\Phi$ -quasioperad. Call a  $p$ -cocartesian edge of  $X^\otimes$  that covers an inert morphism of  $\Lambda(\Phi)$  *inert*. Dually, call any edge of  $X^\otimes$  that covers an active morphism of  $\Lambda(\Phi)$  *active*.

8.12. **Proposition.** *For any perfect operator category and any  $\Phi$ -quasioperad  $X^\otimes$ , the inert morphisms and the active morphisms determine a factorization system on  $X^\otimes$ .*

As in [13, §2.1.4], we can introduce a model category of  $\infty$ -preoperads over a perfect operator category  $\Phi$  whose fibrant objects are  $\infty$ -operads over  $\Phi$ .

8.13. **Notation.** As in [13, Nt. 2.1.4.5], for any operator category  $\Phi$  and any  $\infty$ -operad  $X^\otimes$  over  $\Phi$ , denote by  $X^{\otimes, \natural}$  the object  $(X^\otimes, E)$  of  $s\mathbf{Set}^+_{/N\Lambda(\Phi)}$ , where  $E$  denotes the collection of inert morphisms of  $X^\otimes$ .

8.14. For any perfect operator category  $\Phi$ , consider the following *categorical pattern* (in the sense of Lurie; see [13, App. B])

$$\mathcal{P} = (M, T, \{p_\alpha: \Lambda_0^2 \rightarrow N\Lambda(\Phi)\}_{\alpha \in A})$$

on the simplicial set  $N\Lambda(\Phi)$ . The class  $M$  consists of all the inert morphisms of  $\Lambda(\Phi)$ ; the class  $T$  is the class of all 2-simplices; and the set  $A$  is the set of diagrams  $I \leftarrow J \rightarrow I'$  of  $\Lambda(\Phi)$  in which both  $J \rightarrow I$  and  $J \rightarrow I'$  are inert, and

$$|J| = |J \times_{TI} I| \sqcup |J \times_{TI'} I'| \cong |I| \sqcup |I'|.$$

Now applying [13, Th. B.0.19] to the categorical pattern  $\mathcal{P}$ , one deduces the following.

8.15. **Theorem.** *Suppose  $\Phi$  a perfect operator category. There exists a left proper, tractable, simplicial model structure — called the **operadic model structure** — on  $s\mathbf{Set}^+_{/N\Lambda(\Phi)}$  with the following properties.*

(8.15.1) *A marked map  $X \rightarrow Y$  over  $N\Lambda(\Phi)$  is a cofibration just in case it is a monomorphism.*

(8.15.2) *A marked simplicial set over  $N\Lambda(\Phi)$  is fibrant just in case it is of the form  $Z^{\otimes, \natural}$  for some  $\Phi$ -quasioperad  $Z^\otimes$ .*

(8.15.3) A marked map  $X \rightarrow Y$  over  $N\Lambda(\Phi)$  is a weak equivalence just in case, for any  $\Phi$ -quasioperad  $Z^\otimes$ , the induced map

$$\mathrm{Map}_{/N\Lambda(\Phi)}(Y, Z^{\otimes, \natural}) \rightarrow \mathrm{Map}_{/N\Lambda(\Phi)}(X, Z^{\otimes, \natural})$$

is a weak equivalence.

**8.16. Notation.** For any perfect operator category  $\Phi$ , denote by  $\mathbf{Operad}_\infty^{\Phi, \Delta}$  the simplicial subcategory of  $s\mathbf{Set}_{/N\Lambda(\Phi)}^+$  spanned by the fibrant objects for the operadic model structure. Since this is a fibrant simplicial category, we may apply the nerve to obtain an  $\infty$ -category  $\mathbf{Operad}_\infty^\Phi$ .

Using Lurie's characterization [13, Lm. B.2.4(3)] of  $\mathcal{P}$ -equivalences between fibrant objects, we obtain the following.

**8.17. Proposition.** *Suppose  $\Phi$  a perfect operator category. Then a morphism of  $\Phi$ -quasioperads  $g: X \rightarrow Y$  is an equivalence just in case the following conditions are satisfied.*

(8.17.1) Essential surjectivity. *The functor  $X_{\{1\}} \rightarrow Y_{\{1\}}$  is essentially surjective.*

(8.17.2) Full faithfulness. *For any object  $I \in \Phi$ , any vertex  $x \in X_I$ , and any vertex  $y \in Y_{\{1\}}$ , the induced map*

$$\mathrm{Map}_X^\alpha(x, y) \rightarrow \mathrm{Map}_Y^\alpha(g(x), g(y))$$

is an equivalence, where  $\alpha$  is the unique active morphism  $I \rightarrow \{1\}$  of  $\Lambda(\Phi)$ .

We may apply [13, Pr. B.2.9] to the functors  $\Lambda(\Psi) \rightarrow \Lambda(\Phi)$  induced by operator morphisms  $\Psi \rightarrow \Phi$  thanks to Pr. 8.6; it follows that the operadic model structure on  $s\mathbf{Set}_{/N\Lambda(\Phi)}^+$  enjoys the same functoriality in  $\Phi$  that is enjoyed by the operadic model structure on the category  $s\mathbf{Set}_{/N\Delta_\Phi^{\mathrm{op}}}$  (Pr. 2.13):

**8.18. Proposition.** *For any operator morphism  $G: \Psi \rightarrow \Phi$ , the adjunction*

$$G!: s\mathbf{Set}_{/N\Lambda(\Psi)}^+ \rightleftarrows s\mathbf{Set}_{/N\Lambda(\Phi)}^+ : G^*$$

is a Quillen adjunction for the operadic model structure.

## 9. BOARDMAN–VOGT TENSOR PRODUCTS AND $\infty$ -ALGEBRAS

We can define an analogue of the Boardman–Vogt tensor product introduced in §4 and study its interaction with the model structure introduced in the previous section. Once again, in most cases the proofs are trivial extensions of the proofs of loc. cit.

**9.1. Notation.** For any perfect operator categories  $\Phi$  and  $\Psi$ , define a natural transformation

$$\omega: \wr \circ (T_\Psi \times T_\Phi) \rightarrow T_{\Psi \wr \Phi} \circ \wr$$

as follows. For any pair  $(K, I) \in \Psi \times \Phi$ , let

$$\omega_{(K, I)} = ((\phi_j)_{j \in |T_\Phi I|}, \mathrm{id}_{T_\Phi I}): T_\Psi K \wr T_\Phi I \rightarrow T_{\Psi \wr \Phi}(K \wr I)$$

be the morphism in which

$$\phi_j = \begin{cases} \mathrm{id}_{T_\Psi K} & \text{if } j \in |I| \subset |T_\Phi I|; \\ ! & \text{if } j \notin |I|. \end{cases}$$

Using this, we obtain an induced functor  $W: \Lambda(\Psi) \times \Lambda(\Phi) \longrightarrow \Lambda(\Psi \wr \Phi)$  on the Leinster categories given by the assignment  $(K, I) \mapsto K \wr I$ .

**9.2. Definition.** Suppose  $\Phi$  and  $\Psi$  operator categories, suppose  $X \in s\mathbf{Set}_{/N\Lambda(\Phi)}^+$ , and suppose  $Y \in s\mathbf{Set}_{/N\Lambda(\Psi)}^+$ . Write  $Y^{\Psi \otimes \Phi} X$  for the product  $Y \times X$ , regarded as a marked simplicial set over  $N\Lambda(\Psi \wr \Phi)$  via

$$Y \times X \longrightarrow N\Lambda(\Psi) \times N\Lambda(\Phi) \xrightarrow{W} N\Lambda(\Psi \wr \Phi).$$

We call this the *Boardman–Vogt tensor product* of  $Y$  and  $X$ .

**9.3. Proposition.** *For any operator morphisms  $H: \Psi' \longrightarrow \Psi$  and  $G: \Phi' \longrightarrow \Phi$  between perfect operator categories, and for any objects  $Y \in s\mathbf{Set}_{/N\Lambda(\Psi')}$  and  $X \in s\mathbf{Set}_{/N\Lambda(\Phi')}$ , there is a canonical isomorphism*

$$(H \wr G)_!(Y^{\Psi'} \otimes^{\Phi'} X) \cong H_! Y^{\Psi} \otimes^{\Phi} G_! X$$

of simplicial sets over  $N\Lambda(\Psi \wr \Phi)$ .

The Boardman–Vogt tensor product preserves colimits separately in each variable. Consequently, we have the following.

**9.4. Proposition.** *Suppose  $\Phi$  and  $\Psi$  two perfect operator categories, suppose  $X \in s\mathbf{Set}_{/N\Lambda(\Phi)}$ , and suppose  $Y \in s\mathbf{Set}_{/N\Lambda(\Psi)}$ . Then the functors*

$$Y^{\Psi \otimes \Phi} -: s\mathbf{Set}_{/N\Lambda(\Phi)}^+ \longrightarrow s\mathbf{Set}_{/N\Lambda(\Psi \wr \Phi)}^+ \quad \text{and} \quad -^{\Psi \otimes \Phi} X: s\mathbf{Set}_{/N\Lambda(\Psi)}^+ \longrightarrow s\mathbf{Set}_{/N\Lambda(\Psi \wr \Phi)}^+$$

both admits right adjoints.

**9.5. Notation.** For any two perfect operator categories  $\Phi$  and  $\Psi$ , we obtain a pair of functors

$$\mathbf{Alg}_{\infty}^{\Psi, \Psi \wr \Phi}: (s\mathbf{Set}_{/N\Lambda(\Psi)}^+)^{\text{op}} \times s\mathbf{Set}_{/N\Lambda(\Psi \wr \Phi)}^+ \longrightarrow s\mathbf{Set}_{/N\Lambda(\Phi)}^+$$

and

$$\mathbf{Alg}_{\infty}^{\Phi, \Psi \wr \Phi}: (s\mathbf{Set}_{/N\Lambda(\Phi)}^+)^{\text{op}} \times s\mathbf{Set}_{/N\Lambda(\Psi \wr \Phi)}^+ \longrightarrow s\mathbf{Set}_{/N\Lambda(\Psi)}^+$$

such that  $\mathbf{Alg}_{\infty}^{\Psi, \Psi \wr \Phi}(Y, -)$  is right adjoint to  $Y^{\Psi \otimes \Phi} -$  and  $\mathbf{Alg}_{\infty}^{\Phi, \Psi \wr \Phi}(X, -)$  is right adjoint to  $-^{\Psi \otimes \Phi} X$ .

It is clear that these functors, along with the Boardman–Vogt tensor product, comprise an adjunction of two variables  $s\mathbf{Set}_{/N\Lambda(\Psi)}^+ \times s\mathbf{Set}_{/N\Lambda(\Phi)}^+ \longrightarrow s\mathbf{Set}_{/N\Lambda(\Psi \wr \Phi)}^+$ .

The interaction between this variant of the Boardman–Vogt tensor product and the operadic model structure is the most one could hope for. We apply [13, Rk. B.2.5 and Pr. B.2.9] to the functors  $W: \Lambda(\Psi) \times \Lambda(\Phi) \longrightarrow \Lambda(\Psi \wr \Phi)$  to deduce the following.

**9.6. Theorem.** *For any two perfect operator categories  $\Phi$  and  $\Psi$ , the functors  ${}^{\Psi \otimes \Phi}$ ,  $\mathbf{Alg}_{\infty}^{\Psi, \Psi \wr \Phi}$ , and  $\mathbf{Alg}_{\infty}^{\Phi, \Psi \wr \Phi}$  form a Quillen adjunction of two variables for the operadic model structures.*

**9.7.** Consider the operator category  $\mathbf{F}$ . To relate our Boardman–Vogt tensor product to the monoidal structure  $\odot$  constructed in [13], we need only note that the functor  $\wedge: \Gamma^{\text{op}} \times \Gamma^{\text{op}} \longrightarrow \Gamma^{\text{op}}$  of [13] is isomorphic to the composition of the functor  $W: \Lambda(\mathbf{F}) \times \Lambda(\mathbf{F}) \longrightarrow \Lambda(\mathbf{F})$  with the functor  $\Lambda(\mathbf{F} \wr \mathbf{F}) \longrightarrow \Lambda(\mathbf{F})$  induced by the unique operator morphism  $U: \mathbf{F} \wr \mathbf{F} \longrightarrow \mathbf{F}$ . Consequently, we obtain an isomorphism  $U_!(Y^{\mathbf{F}} \otimes^{\mathbf{F}} X) \cong Y \odot X$ .

**9.8. Notation.** Suppose  $\Phi$  and  $\Psi$  two perfect operator categories. For any  $(\Psi \wr \Phi)$ -quasioperad  $Z$ , we will write  $\mathbf{Mon}_\infty^{\Phi, \Psi \wr \Phi}(Z)$  for  $\mathbf{Alg}_\infty^{\Phi, \Psi \wr \Phi}(U_\Phi, Z)$ , and we will write  $\mathbf{Mon}_\infty^{\Phi, \Psi \wr \Phi}(Z)$  for  $\mathbf{Alg}_\infty^{\Psi, \Psi \wr \Phi}(U_\Psi, Z)$ .

**9.9.** Suppose  $\Phi$  a perfect operator category. Note that we have canonical equivalences  $\Phi \wr \{1\} \simeq \Phi \simeq \{1\} \wr \Phi$ , through which the functor  $\mathbf{Alg}_\infty^{\Phi, \Phi \wr \{1\}}$  and  $\mathbf{Alg}_\infty^{\Phi, \{1\} \wr \Phi}$  may be identified. We write

$$\mathbf{Alg}_\infty^\Phi : (s\mathbf{Set}^+_{/N\Lambda(\Phi)})^{\text{op}} \times s\mathbf{Set}^+_{/N\Lambda(\Phi)} \longrightarrow s\mathbf{Set}^+$$

for the common functor. For any marked map  $Y \longrightarrow N\Lambda(\Phi)$  and any  $\infty$ -operad  $Z$  over  $\Phi$ , we will refer to objects of the quasicategory  $\mathbf{Alg}_\infty^\Phi(Y, Z)$  as  $\infty$ -*algebras over  $Y$  in  $Z$* . If  $Y$  is the terminal  $\Phi$ -operad  $U_\Phi$ , then we will refer to  $Y$ -algebras in  $Z$  as  $\infty$ -*monoids over  $\Phi$  in  $Z$* .

The following theorem is proved exactly as in [13, Th. 2.4.4.3].

**9.10. Theorem.** *For any two operator categories  $\Phi$  and  $\Psi$ , the functor  $W$  induces an equivalence  $U_\Psi^\Psi \otimes^\Phi U_\Phi \xrightarrow{\sim} U_{\Psi \wr \Phi}$ .*

**9.10.1. Corollary.** *For any two operator categories  $\Phi$  and  $\Psi$ , one has, for any  $\Psi \wr \Phi$ -operad  $Z$ , canonical equivalences*

$$\mathbf{Mon}_\infty^{\Psi \wr \Phi}(Z) \simeq \mathbf{Mon}_\infty^\Phi(\mathbf{Mon}_\infty^{\Psi, \Psi \wr \Phi}(Z)) \simeq \mathbf{Mon}_\infty^\Psi(\mathbf{Mon}_\infty^{\Phi, \Psi \wr \Phi}(Z)).$$

## 10. COMPLETE SEGAL OPERADS VERSUS QUASIOPERADS

We now compare our two approaches to the theory of weak  $\Phi$ -operads. Here is the main theorem.

**Theorem.** *For any perfect operator category  $\Phi$ , there exist inverse equivalences of  $\infty$ -categories*

$$P^\otimes : \mathbf{Operad}_{\text{CSS}}^\Phi \simeq \mathbf{Operad}_\infty^\Phi : C.$$

We do this by obtaining a homotopy equivalence of relative categories (Th. 10.16).

We begin with the functor  $P^\otimes$ . To construct this functor, we must first relate the Leinster category  $\Lambda(\Phi)$  to the category of  $\Phi$ -sequences  $\Delta_\Phi$ .

**10.1. Notation.** Suppose  $\Phi$  a perfect operator category. For any object  $\mathbf{m} \in \Delta$ , denote by  $\tilde{\mathcal{C}}(\mathbf{m})$  the twisted arrow category of  $\mathbf{m}$ ; this is a poset whose objects are pairs of integers  $(i, j)$  such that  $0 \leq i \leq j \leq m$ , and  $(i', j') \leq (i, j)$  just in case  $i \leq i' \leq j' \leq j$ .

Now for any  $m$ -simplex  $\sigma : \mathbf{m} \longrightarrow N\Lambda(\Phi)$  corresponding to a sequence of morphisms

$$I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_m$$

in  $\Lambda(\Phi)$ , we obtain a functor  $F_\sigma : \tilde{\mathcal{C}}(\mathbf{m})^{\text{op}} \longrightarrow \Delta_\Phi$ , given by the formula

$$F_\sigma(i, j) := [I_0 \times_{TI_j} I_j \longrightarrow I_1 \times_{TI_j} I_j \longrightarrow \cdots \longrightarrow I_i \times_{TI_j} I_j].$$

We write  $\Delta_\Phi^\sigma$  for the colimit of the diagram  $j \circ F_\sigma$ , where  $j$  denotes the Yoneda embedding  $\Delta_\Phi \hookrightarrow \text{Fun}(\Delta_\Phi^{\text{op}}, \mathbf{Set})$ .

The assignment  $\sigma \mapsto \Delta_\Phi^\sigma$  is functorial with respect to the category  $\text{Simp}(\Lambda(\Phi))$ , whose objects are pairs  $(\mathbf{m}, \sigma)$  consisting of an object  $\mathbf{m} \in \Delta$  and an  $m$ -simplex  $\sigma : \mathbf{m} \longrightarrow N\Lambda(\Phi)$  of  $\Lambda(\Phi)$  and whose morphisms  $(\mathbf{m}, \sigma) \longrightarrow (\mathbf{n}, \tau)$  are morphisms

$\eta: \mathbf{m} \rightarrow \mathbf{n}$  of  $\Delta$  and a natural isomorphism  $\sigma \cong \tau \circ \eta$ ; in particular, we obtain a functor

$$\Delta_{\Phi}^*: \iota \text{Fun}(\mathbf{m}, \Lambda(\Phi)) \rightarrow \iota \text{Fun}(\Delta_{\Phi}^{\text{op}}, \mathbf{Set}).$$

**10.2. Construction.** Suppose  $\Phi$  a perfect operator category, and suppose that  $X: \Delta_{\Phi}^{\text{op}} \rightarrow s\mathbf{Set}$  a functor. For any integer  $m \geq 0$  and any  $m$ -simplex

$$\sigma: \mathbf{m} \rightarrow N\Lambda(\Phi),$$

one may define a simplicial set  $X(\Delta_{\Phi}^{\sigma})$  via right Kan extension, so that

$$X(\Delta_{\Phi}^{\sigma}) \cong \lim_{(i,j) \in \bar{\sigma}(\mathbf{m})} X(F_{\sigma}(i,j)).$$

The assignment  $\sigma \mapsto X(\Delta_{\Phi}^{\sigma})$  defines a functor  $\iota \text{Fun}(\mathbf{m}, \Lambda(\Phi))^{\text{op}} \rightarrow s\mathbf{Set}$ . Apply the construction of [12, Df. 3.2.5.2] to this functor to obtain a map

$$\mathbf{P}^{\otimes}(X)_m \rightarrow N\iota \text{Fun}(\mathbf{m}, \Lambda(\Phi))^{\text{op}}.$$

This map is functorial in  $\mathbf{m} \in \Delta^{\text{op}}$ , whence we obtain a morphism of simplicial spaces

$$\mathbf{P}^{\otimes}(X) \rightarrow \mathbf{N}\Lambda(\Phi),$$

where  $\mathbf{N}\Lambda(\Phi)$  is the classifying diagram of the category  $\Lambda(\Phi)$  in the sense of Rezk [17, 3.5] (except that in each degree we are taking the nerve of the opposite groupoid).

**10.3. Proposition.** *Suppose  $\Phi$  a perfect operator category, and suppose that*

$$X: \Delta_{\Phi}^{\text{op}} \rightarrow \mathbf{Kan}$$

*a functor. Assume that  $X$  that is fibrant for the injective model structure on the category  $\text{Fun}(\Delta_{\Phi}^{\text{op}}, s\mathbf{Set})$ , and that  $X$  classifies a complete Segal  $\Phi$ -operad. Then  $\mathbf{P}^{\otimes}(X)$  is a complete Segal space.*

*Proof.* To see that  $\mathbf{P}^{\otimes}(X)$  is Reedy fibrant, consider any square

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & \mathbf{P}^{\otimes}(X)_m \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \lim_{\mathbf{k} \subsetneq \mathbf{m}} \mathbf{P}^{\otimes}(X)_k. \end{array}$$

Unwinding the definitions, we see that a lift  $\Delta^n \rightarrow \mathbf{P}^{\otimes}(X)_m$  amounts to a lift  $\sigma$  of the diagram

$$\begin{array}{ccc} \Lambda_k^n & \longrightarrow & N\iota \text{Fun}(\mathbf{m}, \Lambda(\Phi))^{\text{op}} \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \lim_{\mathbf{k} \subsetneq \mathbf{m}} N\iota \text{Fun}(\mathbf{k}, \Lambda(\Phi))^{\text{op}} \end{array}$$

along with a compatible collection of maps  $\Delta^J \rightarrow X(\Delta_{\Phi}^{\sigma(\max J)})$ , one for each nonempty subset  $J \subset \mathbf{m}$ . The existence of the lift  $\sigma$  follows from [17, Lm. 3.9], and the compatible collection of maps follows from the injective fibrancy.

The Segal conditions on  $\mathbf{P}^{\otimes}(X)$  reduce to showing that for any  $m$ -simplex  $\sigma: \mathbf{m} \rightarrow \Lambda(\Phi)$ , the induced map

$$X(\Delta_{\Phi}^{\sigma}) \rightarrow X(\Delta_{\Phi}^{\sigma\{0,1\}}) \times_{X(\Delta_{\Phi}^{\sigma\{1\}})} \cdots \times_{X(\Delta_{\Phi}^{\sigma\{m-1\}})} X(\Delta_{\Phi}^{\sigma\{m-1,m\}})$$

is a weak equivalence. But this follows from the fact that, for  $X$  itself, the maps  $s_{\mathbf{n},J}$  are all equivalences.

The completeness conditions on  $\mathbf{P}^\otimes(X)$  reduces to the assertion that for any object  $I \in \Phi$ , the natural map

$$X[I] \longrightarrow X[I = I = I = I] \times_{X[I=I] \times X[I=I]} X[I]$$

is a weak equivalence, where the maps  $X[I = I = I = I] \longrightarrow X[I = I]$  are given by the inclusions  $\{0, 2\} \hookrightarrow \{0, 1, 2, 3\}$  and  $\{1, 3\} \hookrightarrow \{0, 1, 2, 3\}$ . But this follows from the case when  $I = \{1\}$  along with the decomposition

$$X[I] \simeq \prod_{i \in |I|} X[\{i\}]. \quad \square$$

The results of [10] thus motivate the following.

**10.4. Definition.** Suppose  $\Phi$  a perfect operator category, and suppose

$$X: \Delta_\Phi^{\text{op}} \longrightarrow s\mathbf{Set}$$

a functor. Then we define a simplicial set  $P^\otimes(X)$  in the following manner. For any  $\mathbf{m} \in \Delta^{\text{op}}$ , an  $m$ -simplex  $(\sigma, x)$  of  $P^\otimes(X)$  consists of a functor  $\sigma: \mathbf{m} \longrightarrow \Lambda(\Phi)$  and a vertex  $x \in X(\Delta_\Phi^\sigma)_0$ . This is obviously functorial in  $\mathbf{m}$ , and the assignment  $(\sigma, x) \longmapsto \sigma$  defines a projection  $P^\otimes(X) \longrightarrow N\Lambda(\Phi)$ . Hence we obtain a functor  $P^\otimes: \text{Fun}(\Delta_\Phi^{\text{op}}, s\mathbf{Set}) \longrightarrow s\mathbf{Set}/N\Lambda(\Phi)$ .

**10.5.** Suppose  $\Phi$  a perfect operator category, and suppose

$$X: \Delta_\Phi^{\text{op}} \longrightarrow s\mathbf{Set}$$

a functor. Note that the fiber of  $P^\otimes(X)$  over an object  $I \in \Lambda(\Phi)$  is the simplicial set whose  $m$ -simplices are the vertices of  $X(m, I)$ , where  $(m, I)$  denotes the constant sequence  $I = I = \cdots = I$  of length  $m$ . In particular, the fiber over  $\{1\}$  is the simplicial set whose  $m$ -simplices are the vertices of the simplicial set  $(X|\Delta^{\text{op}})_m$ .

**10.6. Proposition.** *Suppose  $\Phi$  a perfect operator category, and suppose*

$$X: \Delta_\Phi^{\text{op}} \longrightarrow s\mathbf{Set}$$

*a functor. Assume that  $X$  is fibrant for the operadic model structure on the category  $\text{Fun}(\Delta_\Phi^{\text{op}}, s\mathbf{Set})$ . Then  $P^\otimes(X)$  is a  $\Phi$ -quasioperad.*

*Proof.* Since the  $m$ -simplices of  $P^\otimes(X)$  can be identified with the 0-simplices of  $\mathbf{P}^\otimes(X)_m$ , Pr 10.3 and [10, Th. 4.11] together imply that  $P^\otimes(X)$  is a quasicategory, whence by [12, Pr. 2.3.1.5],  $P^\otimes(X) \longrightarrow \Lambda(\Phi)$  is an inner fibration.

Suppose  $J \longrightarrow I$  an inert morphism of  $\Lambda(\Phi)$ , and suppose that  $x \in P^\otimes(X)_J$  an object; hence  $x \in X[J]_0$ . Let  $x' \in X[J \times_{TI} I]$  be its image under the map induced by the interval inclusion  $J \times_{TI} I \hookrightarrow J$ . Now we may use the inverse to the isomorphism  $J \times_{TI} I \longrightarrow I$  to define a morphism of  $\Phi$ -sequences

$$[J \times_{TI} I \longrightarrow I] \longrightarrow [J \times_{TI} I],$$

and one may consider the image  $x''$  of  $x'$  in  $X[J \times_{TI} I \longrightarrow I]$  under the induced map. The pair  $(x, x'') \in X[J] \times_{X[J \times_{TI} I]} X[J \times_{TI} I \longrightarrow I]$  is now a cocartesian edge covering  $\phi$ .

Suppose that the following are given:

- two objects  $I, J \in \Phi$ ,
- vertices  $x \in P^\otimes(X)_I$  and  $y \in P^\otimes(X)_J$ ,

- a morphism  $\phi: J \rightarrow I$  of  $\Lambda(\Phi)$ , and
- $p$ -cocartesian edges  $\{y \rightarrow y_i \mid i \in |I|\}$  lying over the inert morphisms  $\{\rho_i: I \rightarrow \{i\} \mid i \in |I|\}$ .

Now the simplicial set  $\text{Map}_{P^\otimes(X)}^\phi(x, y)$  can be identified with the fiber of the natural map

$$X[J] \times_{X[J \times_{TI} I]} X[J \times_{TI} I \rightarrow I] \rightarrow X[J] \times X[I]$$

over the point  $(x, y)$ . The fact that the induced map

$$\text{Map}_{X^\otimes}^\phi(x, y) \rightarrow \prod_{i \in |I|} \text{Map}_{X^\otimes}^{\rho_i \circ \phi}(x, y_i)$$

is an equivalence thus follows from the decomposition

$$\begin{aligned} X[J] \times_{X[J \times_{TI} I]} X[J \times_{TI} I \rightarrow I] &\simeq X[J] \times_{\prod_{i \in |I|} X[J_i]} \prod_{i \in |I|} X[J_i \rightarrow \{i\}] \\ &\rightarrow X[J] \times \prod_{i \in |I|} X[\{i\}] \\ &\simeq X[J] \times X[I]. \end{aligned}$$

Finally, we must show that for any object  $I \in \Phi$  and for any  $I$ -tuple

$$(y_i)_{i \in |I|} \in \prod_{i \in |I|} P^\otimes(X)_{\{i\}},$$

there exists an object  $y \in P^\otimes(X)_I$  a collection of cocartesian edges  $y \rightarrow y_i$  lying over the inert edges  $I \rightarrow \{i\}$ . For this, choose  $y \in X[I]$  that maps to  $(y_i)_{i \in |I|}$ . (This is possible since the map  $X[I] \rightarrow \prod_{i \in |I|} X[\{i\}]$  is a trivial fibration.) For any point  $i \in |I|$ , the image of  $y$  under the natural map

$$X[I] \cong X[I] \times_{X[\{i\}]} X[\{i\}] \rightarrow X[I] \times_{X[\{i\}]} X[\{i\} = \{i\}]$$

is the desired cocartesian edge  $y \rightarrow y_i$ .  $\square$

**10.7. Example.** When  $\Phi = \{1\}$ , the functor  $P^\otimes$  carries a complete Segal space  $X$  to the quasicategory whose  $m$ -simplices are the 0-simplices of  $X_m$ . By a theorem of Joyal and Tierney [10, Th. 4.11], this is known to be an equivalence of homotopy theories.

It is obvious from the construction given here that  $P^\otimes$  is compatible with changes of operator category, in the following sense.

**10.8. Proposition.** *Suppose  $G: \Psi \rightarrow \Phi$  is an operator morphism between perfect operator categories. Then we have a natural isomorphism of functors*

$$P^\otimes \circ G^* \cong G^* \circ P^\otimes.$$

**10.9. Proposition.** *Suppose  $\Phi$  a perfect operator category. A morphism  $X \rightarrow Y$  between injectively fibrant complete Segal  $\Phi$ -operads is a weak equivalence just in case the induced morphism  $P^\otimes(X) \rightarrow P^\otimes(Y)$  is so.*

*Proof.* Let us use the characterizations of Prs. 2.8 and 8.17. Combining the example and the proposition above, we deduce that a morphism  $X \rightarrow Y$  of complete Segal  $\Phi$ -operads induces an essentially surjective functor on underlying complete Segal spaces just in case the morphism  $P^\otimes(X) \rightarrow P^\otimes(Y)$  induces an essentially surjective functor on underlying quasicategories.

On the other hand, if  $I \in \Phi$ , and if  $\alpha: I \rightarrow \{1\}$  is the morphism of  $\Lambda(\Phi)$  given by  $I \rightarrow \{t\} \rightarrow T$ , then for any complete Segal  $\Phi$ -operad  $X$ , the mapping space  $\text{Map}_X^\alpha(x, y)$  is equivalent to the fiber of the map

$$X[I \rightarrow \{1\}] \rightarrow X[I] \times X[\{1\}].$$

In particular, a morphism  $X \rightarrow Y$  of complete Segal  $\Phi$ -operads is fully faithful just in case the induced morphism  $P^\otimes(X) \rightarrow P^\otimes(Y)$  of  $\Phi$ -quasicategories is so.  $\square$

In other words,  $P^\otimes$  is a relative functor in the sense of [4], and it reflects weak equivalences.

To define a functor in the opposite direction, we introduce the following.

**10.10. Construction.** Suppose  $\Phi$  a perfect operator category, and suppose  $(\mathbf{m}, I)$  a  $\Phi$ -sequence. Define a poset  $A(\mathbf{m}, I)$  as follows: the objects are triples  $(r, s, i)$ , where  $r$  and  $s$  are integers such that  $0 \leq r \leq s \leq m$ , and  $i \in |I_s|$ ; we declare that  $(r, s, i) \leq (r', s', i')$  just in case  $r \leq r' \leq s' \leq s$ , and  $i' \mapsto i$  under the map  $|I_{s'}| \rightarrow |I_s|$ .

Let  $A(\mathbf{m}, I)^\dagger$  be the set of morphisms of  $A(\mathbf{m}, I)$  of the form  $(r, s, i) \rightarrow (r, s', i')$ . Using the factorization system on  $\Lambda(\Phi)$  of Lm. 8.3, we deduce that there is a functor  $A(\mathbf{m}, I) \rightarrow \Lambda(\Phi)$  given by the assignment

$$(r, s, i) \mapsto I_{r,i} := I_r \times_{I_s} \{i\}$$

under which morphisms of  $A(\mathbf{m}, I)^\dagger$  are carried to inert morphisms of  $\Lambda(\Phi)$ .

Now the assignment  $(\mathbf{m}, I) \rightarrow (NA(\mathbf{m}, I), A(\mathbf{m}, I)^\dagger)$  defines a functor

$$NA: \Delta_\Phi \rightarrow s\mathbf{Set}_{/N\Lambda(\Phi)}^+.$$

For any object  $X^\otimes \in s\mathbf{Set}_{/N\Lambda(\Phi)}^+$ , write  $C(X)$  for the functor  $\Delta_\Phi^{\text{op}} \rightarrow s\mathbf{Set}$  given by the assignment

$$(\mathbf{m}, I) \mapsto \text{Map}_{N\Lambda(\Phi)}^\sharp(NA(\mathbf{m}, I), X^\otimes).$$

**10.11. Example.** When  $\Phi = \{1\}$ , the category  $A(\mathbf{m}, \{1\})$  is the opposite of the twisted arrow category  $\tilde{\mathcal{O}}(\mathbf{m})$ , and in  $N\tilde{\mathcal{O}}(\mathbf{m})^{\text{op}}$ , the marked edges are precisely the morphisms  $(r, s) \rightarrow (r, s')$ . Now the inclusion  $\mathbf{m} \hookrightarrow N\tilde{\mathcal{O}}(\mathbf{m})^{\text{op}}$  given by  $r \mapsto (r, m)$  induces a marked anodyne morphism on nerves. Consequently, for any quasicategory  $X$ , one has  $C(X)_m \simeq \iota \text{Fun}(\Delta^m, X)$ , which is an inverse homotopy equivalence of  $P^\otimes$  by the theorem of Joyal and Tierney [10, Th. 4.11].

The construction  $(m, I) \mapsto NA(m, I)$  is compatible with changes of operator category. That is, for any operator morphism  $G: \Psi \rightarrow \Phi$  between perfect operator categories, one has a canonical isomorphism  $G_! NA(m, I) \cong NA(m, GI)$ . Consequently, we have the following.

**10.12. Lemma.** *For any operator morphism  $G: \Psi \rightarrow \Phi$  between perfect operator categories, there is a natural isomorphism*

$$C \circ G^* \cong G^* \circ C.$$

**10.13. Proposition.** *Suppose  $\Phi$  a perfect operator category, and suppose  $X^\otimes$  a  $\Phi$ -quasioperad. Then  $C(X^\otimes)$  is fibrant for the operadic model structure.*

*Proof.* The fact that  $C(X^\otimes)$  is fibrant for the injective model structure follows from [12, Rk. A.2.9.27] and the observation that  $(m, I) \mapsto (NA(m, I), A(m, I)^\dagger)$  is cofibrant for the projective model structure on  $\text{Fun}(\Delta_\Phi, s\mathbf{Set}_{/NA(\Phi)}^+)$ .

For any  $\Phi$ -sequence  $[I_0 \rightarrow \cdots \rightarrow I_m]$ , the canonical map

$$\coprod_{i \in |I_m|} NA[I_{0,i} \rightarrow \cdots \rightarrow \{i\}] \rightarrow NA[I_0 \rightarrow \cdots \rightarrow I_m]$$

is an isomorphism by definition. Moreover, for any integer  $0 \leq k \leq m$ , an elementary computation shows that the inclusion

$$NA[I_0 \rightarrow \cdots \rightarrow I_k] \cup^{NA[I_k]} NA[I_k \rightarrow \cdots \rightarrow I_m] \hookrightarrow NA[I_0 \rightarrow \cdots \rightarrow I_m]$$

is  $\mathcal{P}$ -anodyne.

It thus remains only to observe that the underlying simplicial set  $p^*C(X^\otimes)$  is a complete Segal space, and this follows from Lm. 10.12 and Ex. 10.11.  $\square$

**10.14. Proposition.** *Suppose  $\Phi$  a perfect operator category. A morphism  $X^\otimes \rightarrow Y^\otimes$  between  $\Phi$ -quasioperads is a weak equivalence just in case the induced morphism  $C(X^\otimes) \rightarrow C(Y^\otimes)$  of complete Segal  $\Phi$ -operads is so.*

*Proof.* Let us use the characterizations of Prs. 2.8 and 8.17. Applying Lm. 10.12 and Ex. 10.11, we see that  $X^\otimes \rightarrow Y^\otimes$  is essentially surjective just in case the morphism  $C(X^\otimes) \rightarrow C(Y^\otimes)$  is so.

To complete the proof, let us note that for any object  $I \in \Phi$ , the inclusion  $\Delta^1 \hookrightarrow NA[I \rightarrow \{1\}]$  given by  $r \mapsto (r, m, 1)$  is  $\mathcal{P}$ -anodyne. Consequently, we obtain an equivalence

$$[I/\{1\}] \text{Map}_{C(X^\otimes)}((x_i)_{i \in |I|}, y) \simeq \text{Map}_{X^\otimes}^\alpha(x, y)$$

for any objects  $x \in X_I^\otimes$ ,  $y \in X_{\{1\}}^\otimes$ , and any collection  $\{x \rightarrow x_i \mid i \in |I|\}$  of inert morphisms.  $\square$

In other words,  $C$ , when restricted to  $\Phi$ -quasioperads is a relative functor in the sense of [4], and it reflects weak equivalences. In fact, it is part of a Quillen pair.

**10.14.1. Corollary.** *Suppose  $\Phi$  a perfect operator category. Then the functors*

$$B: \text{Fun}(\Delta_\Phi^{\text{op}}, s\mathbf{Set}) \rightleftarrows s\mathbf{Set}_{/NA(\Phi)}^+ : C$$

*given by the formulas*

$$B(X) := \int^{(\mathbf{m}, I)} NA(\mathbf{m}, I) \times X(\mathbf{m}, I)^\sharp$$

*and*

$$C(Y^\otimes)(\mathbf{m}, I) := \text{Map}_{NA(\Phi)}^\sharp(NA(\mathbf{m}, I), Y^\otimes)$$

*form a Quillen pair for the operadic model structures.*

**10.15.** Note that, using the functor  $C$  along with the construction of [13, Cnstr. 2.1.1.7], we can now form the *nerve* of any symmetric operad enriched in Kan complexes as a complete Segal  $\mathbf{F}$ -operad.

We are now ready to state the main result of this section.

10.16. **Theorem.** *For any perfect operator category  $\Phi$ , the functors*

$$P^\otimes: \mathbf{Operad}_{\text{CSS}}^\Phi \longrightarrow \mathbf{Operad}_\infty^\Phi \quad \text{and} \quad C: \mathbf{Operad}_\infty^\Phi \longrightarrow \mathbf{Operad}_{\text{CSS}}^\Phi$$

*are inverse equivalences of  $\infty$ -categories.*

*Proof.* Since we have shown that each functor is conservative, it is sufficient for us to furnish a natural transformation  $\eta: \text{id} \rightarrow P^\otimes \circ C$  that is objectwise essentially surjective and fully faithful.

For any object  $\mathbf{m} \in \Delta$ , any  $m$ -simplex  $\sigma: \mathbf{m} \rightarrow \Lambda(\Phi)$ , and any integers  $0 \leq i \leq j \leq m$ , consider the functor  $f_{\sigma, i, j}: AF_\sigma(i, j) \rightarrow \mathbf{m}$  over  $\Lambda(\Phi)$  given by the assignment  $(r, s, k) \mapsto r$ ; these are compatible with one another and hence define a map

$$f_\sigma: \text{colim}_{(i, j) \in \tilde{\sigma}(\mathbf{m})} NAF_\sigma(i, j) \longrightarrow \Delta^m$$

in  $s\mathbf{Set}_{/N\Lambda(\Phi)}^+$ . This induces, for any  $\Phi$ -quasioperad  $X^\otimes$ , a map

$$X_m^\otimes \longrightarrow \text{Mor}_{N\Lambda}(\text{colim}_{(i, j) \in \tilde{\sigma}(\mathbf{m})} NAF_\sigma(i, j), X^\otimes) \cong P^\otimes(C(X^\otimes))_m,$$

natural in  $X^\otimes$ , hence a natural transformation  $\eta: \text{id} \rightarrow P^\otimes \circ C$ .

When  $\Phi = \{1\}$ , the functor  $f_{\sigma, i, j}$  is the projection  $\tilde{\sigma}(\mathbf{m})^{\text{op}} \rightarrow \mathbf{m}$  given by  $(r, s) \rightarrow r$ , which is a retract of the map considered in Ex. 10.11, which thus must induce an equivalence. Hence by functoriality in  $\Phi$ , it follows that  $\eta$  is objectwise an equivalence on underlying quasicategories, and in particular is objectwise essentially surjective.

Suppose  $I \in \Phi$ , and consider the 1-simplex of  $\Lambda(\Phi)$  given by the unique active morphism  $\alpha: I \rightarrow \{1\}$ . The map  $f_\alpha: NA[I \rightarrow \{1\}] \rightarrow \Delta^1$  is a retract of the inclusion  $\Delta^1 \hookrightarrow NA[I \rightarrow \{1\}]$  considered in the proof of Pr. 10.14. Consequently, it induces equivalences

$$\text{Map}_{X^\otimes}^\alpha(x, y) \xrightarrow{\simeq} \text{Map}_{P^\otimes(C(X^\otimes))}^\alpha(x, y),$$

whence  $\eta$  is objectwise fully faithful.  $\square$

10.17. **Conjecture.** *We expect that the axiom system given in [5] should have an analogue for homotopy theories of weak operads over a fixed operator category  $\Phi$ , and that a corresponding unicity theorem should hold.*

We conclude with a remark on the compatibility between the equivalences

$$\mathbf{Operad}_{\text{CSS}}^\Phi \simeq \mathbf{Operad}_\infty^\Phi$$

and the Boardman–Vogt tensor product.

10.18. **Proposition.** *Suppose  $\Phi$  a perfect operator category, and suppose  $X$  and  $Z$  two  $\Phi$ -quasioperads. Then one has a canonical equivalence*

$$C(\mathbf{Alg}_\infty^\Phi(X, Z)) \simeq \mathbf{Alg}^\Phi(CX, CZ).$$

*Proof.* The claim is equivalent to the claim that for any simplicial space  $Y$ , one has a projection formula

$$B(Y)^{\{1\}} \otimes^\Phi X \simeq B(Y^{\{1\}} \otimes^\Phi CX).$$

Note that  $B$  preserves all weak equivalences (as it is left Quillen for the *injective* model structure on  $\text{Fun}(\Delta_\Phi^{\text{op}}, s\mathbf{Set})$ ) and all colimits; hence it preserves all homotopy colimits. Consequently, we may assume that  $Y$  is either  $\Delta^0$  or  $\Delta^1$ . In the former case, the result is obvious; in the latter, it follows from a computation.  $\square$

In general, we expect that the equivalence  $C$  is fully compatible with the Boardman–Vogt tensor products for any perfect operator categories, so that

$$C(Y^\Psi \otimes^\Phi X) \simeq C(Y)^\Psi \otimes^\Phi C(X).$$

This is true when both  $X$  and  $Y$  are terminal quasioperads by Th. 4.12 and Th. 9.10.

## 11. SOME EXAMPLES OF WEAK $\Phi$ -MONOIDS

We may now prove the assertions stated in the introduction. First, we note that [13, Cor 5.1.1.5] states the following.

**11.1. Proposition.** *The  $\infty$ -operad  $E_\infty$  over  $\mathbf{F}$  is equivalent to the terminal  $\infty$ -operad over  $\mathbf{F}$ .*

Applying the functor  $C$ , we may also state this result in the context of complete Segal  $\mathbf{F}$ -operads.

**11.1.1. Corollary.** *The complete Segal  $\mathbf{F}$ -operad  $C(E_\infty)$  is equivalent to the terminal weak  $\mathbf{F}$ -operad  $U_{\mathbf{F}}$ .*

We may equally well state this result from the perspective of the  $\infty$ -categories of algebras.

**11.1.2. Corollary.** *The  $\infty$ -category of  $E_\infty$ -algebras in an  $\infty$ -operad  $Z$  over  $\mathbf{F}$  is equivalent to the quasicategory associated with the complete Segal space  $\mathbf{Mon}^{\mathbf{F}}(Z)$ .*

Similarly, [13, Pr. 4.1.2.10 and Ex. 5.1.0.7], in light of the theory developed here, state the following.

**11.2. Proposition.** *The associative  $\infty$ -operad over  $\mathbf{F}$  and the  $\infty$ -operad  $E_1$  over  $\mathbf{F}$  are each equivalent to the symmetrization of the terminal  $\infty$ -operad over  $\mathbf{O}$ .*

**11.2.1. Corollary.** *The complete Segal  $\mathbf{F}$ -operad  $C(E_1)$  is equivalent to the symmetrization of the terminal weak  $\mathbf{O}$ -operad  $U_{\mathbf{O}}$ .*

**11.2.2. Corollary.** *The  $\infty$ -category of  $E_1$ -algebras in an  $\infty$ -operad  $Z$  over  $\mathbf{F}$  is equivalent to the quasicategory associated with the complete Segal space  $\mathbf{Mon}^{\mathbf{O}}(u^*Z)$ .*

Since the operad  $A_n$  is the suboperad of  $E_1$  generated by the operations of arity  $\leq n$ , we deduce the following.

**11.3. Proposition.** *The complete Segal operad  $C(A_n)$  is equivalent to the symmetrization of the terminal complete Segal  $\mathbf{O}_{\leq n}$ -operad.*

**11.3.1. Corollary.** *The  $\infty$ -category of  $A_n$ -algebras in an  $\infty$ -operad  $Z$  over  $\mathbf{F}$  is equivalent to the quasicategory associated with the complete Segal space  $\mathbf{Mon}^{\mathbf{O}_{\leq n}}(u^*Z)$ .*

Finally, we have the following, which follows from [13, Th. 5.1.2.2] and 9.7.

**11.4. Proposition.** *For any integer  $k \geq 0$ , the  $\infty$ -operad  $E_k$  over  $\mathbf{F}$  is equivalent to the symmetrization of the terminal  $\infty$ -operad over  $\mathbf{O}^{(k)}$ .*

**11.4.1. Corollary.** *For any integer  $k \geq 0$ , the  $\infty$ -category of  $E_k$ -algebras in an  $\infty$ -operad  $Z$  over  $\mathbf{F}$  is equivalent to the quasicategory associated with the complete Segal space  $\mathbf{Mon}^{\mathbf{O}^{(k)}}(U^*Z)$ .*

11.4.2. **Corollary.** *The  $\infty$ -category of  $E_k$ -algebras (in spaces) is equivalent to the  $\infty$ -category of left fibrations*

$$X \longrightarrow N\Lambda(\mathbf{O}^{(k)}) \simeq N\Theta_k^{\text{op}}$$

*satisfying the Segal condition, so that for any object  $I \in \mathbf{O}^{(k)}$ , the inert morphisms  $\{\rho_i: I \rightarrow \{i\} \mid i \in |I|\}$  induce an equivalence*

$$X_I \xrightarrow{\sim} \prod_{i \in |I|} X_{\{i\}}.$$

11.5. **Example.** The formalism we have introduced invites the study of a wide range of new examples as well. In particular, let us contemplate the operator categories  $\mathbf{O}_{\leq m}^{(n)} := (\mathbf{O}^{(n)})_{\leq m}$ . They fit together in a diagram

$$\begin{array}{ccccccc} \{1\} & \longrightarrow & \mathbf{O}_{\leq 1} & \longrightarrow & (\mathbf{O} \wr \mathbf{O})_{\leq 1} & \longrightarrow \cdots \longrightarrow & \mathbf{O}_{\leq 1}^{(n)} & \longrightarrow \cdots \longrightarrow & \mathbf{F}_{\leq 1} \\ \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{1\} & \longrightarrow & \mathbf{O}_{\leq 2} & \longrightarrow & (\mathbf{O} \wr \mathbf{O})_{\leq 2} & \longrightarrow \cdots \longrightarrow & \mathbf{O}_{\leq 2}^{(n)} & \longrightarrow \cdots \longrightarrow & \mathbf{F}_{\leq 2} \\ \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \vdots & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{1\} & \longrightarrow & \mathbf{O}_{\leq m} & \longrightarrow & (\mathbf{O} \wr \mathbf{O})_{\leq m} & \longrightarrow \cdots \longrightarrow & \mathbf{O}_{\leq m}^{(n)} & \longrightarrow \cdots \longrightarrow & \mathbf{F}_{\leq m} \\ \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \vdots & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{1\} & \longrightarrow & \mathbf{O} & \longrightarrow & \mathbf{O} \wr \mathbf{O} & \longrightarrow \cdots \longrightarrow & \mathbf{O}^{(n)} & \longrightarrow \cdots \longrightarrow & \mathbf{F}. \end{array}$$

By forming the symmetrization  $A_m^n$  of the terminal weak  $\mathbf{O}_{\leq m}^{(n)}$ -operad, we obtain a diagram of weak symmetric operads

$$\begin{array}{ccccccc} E_0 & \longrightarrow & A_1 & \longrightarrow & A_1^2 & \longrightarrow \cdots \longrightarrow & A_1^n & \longrightarrow \cdots \longrightarrow & A_1^\infty \\ \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E_0 & \longrightarrow & A_2 & \longrightarrow & A_2^2 & \longrightarrow \cdots \longrightarrow & A_2^n & \longrightarrow \cdots \longrightarrow & A_2^\infty \\ \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \vdots & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E_0 & \longrightarrow & A_m & \longrightarrow & A_m^2 & \longrightarrow \cdots \longrightarrow & A_m^n & \longrightarrow \cdots \longrightarrow & A_m^\infty \\ \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \vdots & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \parallel & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow \cdots \longrightarrow & E_n & \longrightarrow \cdots \longrightarrow & E_\infty. \end{array}$$

This is an interesting bifiltration  $\{A_m^n\}_{m,n}$  of the  $E_\infty$  operad that incorporates both the  $E_n$  operads as well as the  $A_n$  operads. This filtration appears to include much that is already known about obstruction theories for finding  $E_\infty$  structures

on spectra. In particular, when  $n = 1$ , we are simply looking at the filtration of  $E_1 \simeq A_\infty$  by the operads  $A_n$ . When  $n = \infty$ , we are filtering  $E_\infty$  by the suboperads generated by the operations of arity  $\leq m$ . When  $m = \infty$ , we are looking at the filtration of  $E_\infty$  by the operads  $E_n$ . And finally, when  $m = n + 1$ , we expect that an algebra over  $A_{n+1}^n$  is the same thing as an  $n$ -stage  $E_\infty$  structure in the sense of Robinson [18, §5.2], thereby giving his “diagonal” filtration on  $E_\infty$ , though we have not checked this.

#### APPENDIX A. A PROOF OF TH. 4.12

Suppose  $\Phi$  and  $\Psi$  two operator categories. The claim of the theorem is that the functor

$$W: N\Delta_\Psi^{\text{op}} \times_{N\Delta^{\text{op}}} N\Delta_\Phi^{\text{op}} \longrightarrow N\Delta_{\Psi\wr\Phi}^{\text{op}}$$

is a weak equivalence in the operadic model structure on  $s\mathbf{Set}/_{N\Delta_{\Psi\wr\Phi}^{\text{op}}}$ . Note that this functor is faithful (in fact, pseudomonc), but not full. Let us call any object or morphism in its image *rectangular*.

The proof proceeds as follows: first we find a full subcategory  $\Delta_\Psi^{\text{op}} \times_{\Delta^{\text{op}}} \Delta_\Phi^{\text{op}} \subset V^{\text{op}} \subset \Delta_{\Psi\wr\Phi}^{\text{op}}$ , and we show that the inclusion  $N\Delta_\Psi^{\text{op}} \times_{N\Delta^{\text{op}}} N\Delta_\Phi^{\text{op}} \hookrightarrow NV^{\text{op}}$  is an operadic weak equivalence. Then we replace the inclusion  $NV^{\text{op}} \hookrightarrow N\Delta_{\Psi\wr\Phi}^{\text{op}}$  by a suitable left fibration  $Y \rightarrow N\Delta_{\Psi\wr\Phi}^{\text{op}}$ , which we then show has contractible fibers by showing that every map in from a finite simplicial set can, up to homotopy, be “coned off.”

A.1. Now let  $V$  denote the full subcategory of  $\Delta_{\Psi\wr\Phi}^{\text{op}}$  spanned by those objects  $[K_0 \rightarrow \cdots \rightarrow K_m]$  such that for every integer  $1 \leq i \leq m$ , the fibers of the map  $K_{i-1} \rightarrow K_i$  are all rectangular. Note that if  $Z$  is a  $\Psi\wr\Phi$ -operad, then for any object  $[K_0 \rightarrow \cdots \rightarrow K_m] \in V$ , we have a natural equivalence from  $Z[K_0 \rightarrow \cdots \rightarrow K_m]$  to the (homotopy) limit of a diagram

$$\begin{array}{ccc} \prod_{k_1 \in |K_1|} Z[K_{0,k_1} \rightarrow \{k_1\}] & \cdots & \prod_{k_m \in |K_m|} Z[K_{m-1,k_m} \rightarrow \{k_m\}], \\ \swarrow & & \swarrow \\ \prod_{k_1 \in |K_1|} Z[\{k_1\}] & & \prod_{k_{m-1} \in |K_{m-1}|} Z[\{k_{m-1}\}] \end{array}$$

in which all the terms are the fiber of  $Z$  over rectangular objects, and all the maps that appear are induced by rectangular morphisms. Consequently, one may extend any map of simplicial sets  $N\Delta_\Psi^{\text{op}} \times_{N\Delta^{\text{op}}} N\Delta_\Phi^{\text{op}} \rightarrow Z$  over  $N\Delta_{\Psi\wr\Phi}^{\text{op}}$ , in a unique way up to homotopy, to a map  $NV^{\text{op}} \rightarrow Z$  over  $N\Delta_{\Psi\wr\Phi}^{\text{op}}$ ; in other words, the map

$$[NV^{\text{op}}, Z]_{N\Delta_{\Psi\wr\Phi}^{\text{op}}} \longrightarrow [N\Delta_\Psi^{\text{op}} \times_{N\Delta^{\text{op}}} N\Delta_\Phi^{\text{op}}, Z]_{N\Delta_{\Psi\wr\Phi}^{\text{op}}}$$

is a bijection. Hence the functor  $W$  induces a weak equivalence

$$N\Delta_\Psi^{\text{op}} \times_{N\Delta^{\text{op}}} N\Delta_\Phi^{\text{op}} \xrightarrow{\simeq} NV^{\text{op}}$$

in the operadic model structure on  $s\mathbf{Set}/_{N\Delta_{\Psi\wr\Phi}^{\text{op}}}$ .

A.2. Now the inclusion  $NV^{\text{op}} \hookrightarrow (N\Delta_{\Psi\wr\Phi}^{\text{op}})_{NV^{\text{op}}/}$  is a trivial cofibration for the co-variant model structure, and the target is fibrant [13, Cor. 2.1.2.2]. Hence it suffices now to prove that the natural map of left fibrations  $(N\Delta_{\Psi\wr\Phi}^{\text{op}})_{NV^{\text{op}}/} \rightarrow N\Delta_{\Psi\wr\Phi}^{\text{op}}$

is a pointwise weak equivalence. In other words, we aim to show that for any object  $(\mathbf{m}, K) = [K_0 \rightarrow \cdots \rightarrow K_m]$ , the nerve of the ordinary category  $V_{(\mathbf{m}, K)}/$  is contractible.

A.3. The objects of  $V_{(\mathbf{m}, K)}/$  can be described as triples

$$(\eta: \mathbf{m} \rightarrow \mathbf{n}, L: \mathbf{n} \rightarrow \Psi \wr \Phi, \phi: K \rightarrow L \circ \eta)$$

consisting of a morphism  $\eta$  of  $\Delta$ , a functor  $L: \mathbf{n} \rightarrow \Psi \wr \Phi$  such that every fiber of each morphism  $L_{j-1} \rightarrow L_j$  is rectangular, and a map  $\phi: K \rightarrow L \circ \eta$  such that each morphism  $\phi_i: K_i \hookrightarrow L_{\eta(i)}$  is an interval inclusion and the squares

$$\begin{array}{ccc} K_{i-1} & \hookrightarrow & L_{\eta(i-1)} \\ \downarrow & & \downarrow \\ K_i & \hookrightarrow & L_{\eta(i)} \end{array}$$

are all pullbacks. Now denote by  $R_{(\mathbf{m}, K)} \subset V_{(\mathbf{m}, K)}/$  the full subcategory spanned by those objects  $(\eta: \mathbf{m} \rightarrow \mathbf{n}, L, \phi)$ , such that:

- $\eta$  has the property that  $\eta(0) = 0$  and  $\eta(m) = n$ ,
- $L_{\eta(i)} = K_i$  for any  $0 \leq i \leq m$ , and
- $\phi_i$  is the identity for any  $0 \leq i \leq m$ .

The inclusion  $i$  of this subcategory admits a retraction  $r$ , given by the functor that carries any triple  $(\theta: \mathbf{m} \rightarrow \mathbf{p}, M, \psi) \in V_{(\mathbf{m}, K)}/$  to the triple  $(\eta: \mathbf{m} \rightarrow \mathbf{n}, L, \phi) \in R_{(\mathbf{m}, K)}$  in which:

- $\eta: \mathbf{m} \rightarrow \mathbf{n} = \mathbf{p}_{\theta(0)}/_{/\theta(m)}$  is the morphism induced by  $\theta$ ,
- $L_j = M_j \times_{M_{\theta(m)}} K_m$  for any  $\theta(0) \leq j \leq \theta(m)$ , and
- $\phi_i$  is the identity for any  $0 \leq i \leq m$ .

The factorization of  $\psi$  through  $\phi$  provides a natural transformation  $i \circ r \rightarrow \text{id}$ ; hence we have a weak homotopy equivalence  $NV_{(\mathbf{m}, K)}/ \simeq NR_{(\mathbf{m}, K)}$ . It thus remains to show that the nerve of  $R_{(\mathbf{m}, K)}$  is contractible.

A.4. If there exists an integer  $1 \leq i \leq m$  such that the morphism  $K_{i-1} \rightarrow K_i$  is an isomorphism, then it is easy to see that the natural functor

$$R_{[K_0 \rightarrow \cdots \rightarrow K_{i-1} \rightarrow K_{i+1} \rightarrow \cdots \rightarrow K_m]} \rightarrow R_{[K_0 \rightarrow \cdots \rightarrow K_m]}$$

induces an equivalence on nerves. Consequently, we may assume that none of the morphisms  $K_{i-1} \rightarrow K_i$  is an isomorphism, and consequently, that for any object  $(\eta: \mathbf{m} \rightarrow \mathbf{n}, L, \phi) \in R_{(\mathbf{m}, K)}$ , the map  $\eta$  is injective.

Now for any integer  $m \geq 0$ , we note that the natural functor

$$R_{[K_0 \rightarrow \cdots \rightarrow K_m]} \rightarrow R_{[K_0 \rightarrow K_1]} \times \cdots \times R_{[K_{m-1} \rightarrow K_m]}$$

is an equivalence of categories. Hence we reduce to the case in which  $m = 1$ .

A.5. So suppose that  $m = 1$ , and let us set about showing that the nerve of the category  $R_{[K_0 \rightarrow K_1]}$  is contractible. By definition any morphism  $g: L \rightarrow M$  of  $\Psi \wr \Phi$  can be factored, in an essentially unique manner, as a map  $L \rightarrow M(g)$  covering the identity morphism in  $\Phi$ , followed by a morphism  $M(g) \rightarrow M$  that is cocartesian for the coronal fibration  $\Psi \wr \Phi \rightarrow \Phi$ . Let us call this a **coronal factorization**. The resulting sequence  $[L \rightarrow M(g) \rightarrow M]$  clearly lies in  $V$ .

Fix a coronal factorization  $[K_0 \rightarrow K_{01} \rightarrow K_1] \in R_{[K_0 \rightarrow K_1]}$ . Our aim is now to produce, for any finite simplicial set  $X$  and any map  $g: X \rightarrow NR_{[K_0 \rightarrow K_1]}$ ,

a weakly contractible simplicial set  $P_g^* \supset X$  with a distinguished vertex  $v$  and an extension of  $g$  to a map  $G: P_g^* \rightarrow NR_{[K_0 \rightarrow K_1]}$  such that  $G|_X = g$  and  $G(v) = [K_0 \rightarrow K_{01} \rightarrow K_1] \in R_{[K_0 \rightarrow K_1]}$ . This will complete the proof.

A.6. We now set about constructing the simplicial set  $P_g^*$ . Denote by  $\mathbf{I}$  the category whose objects are objects  $\mathbf{n} \in \Delta$  such that  $n \geq 1$  and whose morphisms  $\mathbf{n} \rightarrow \mathbf{n}'$  are morphisms  $\mathbf{n} \rightarrow \mathbf{n}'$  of  $\Delta$  that carry 0 to 0 and  $n$  to  $n'$ . We have an obvious projection  $R_{[K_0 \rightarrow K_1]} \rightarrow \mathbf{I}$  given by  $(\eta: \mathbf{1} \rightarrow \mathbf{n}, L, \phi) \mapsto \mathbf{n}$ .

Let  $\mathbf{P}$  be the following category. The objects will be triples  $(\mathbf{n}, \mathbf{p}, h)$ , where  $\mathbf{m}, \mathbf{p} \in \mathbf{I}$ , and  $h$  is a functor  $\mathbf{p} \rightarrow \text{Fun}(\mathbf{1}, \mathbf{n})$  such that  $h(0) = 00$ ,  $h(m) = nn$ , and for any  $1 \leq i \leq n$ , the map  $h(i-1) \rightarrow h(i)$  is of the form of one of the following:

$$ij \rightarrow ik, \quad ik \rightarrow jk, \quad \text{or} \quad ik \rightarrow (i+1)(k+1).$$

A morphism  $(\mathbf{n}, \mathbf{p}, h) \rightarrow (\mathbf{n}', \mathbf{p}', h')$  of  $\mathbf{P}$  is a morphism  $\sigma: \mathbf{n} \rightarrow \mathbf{n}'$  of  $\mathbf{I}$  such that one has an inclusion

$$\{h(i) \mid 0 \leq i \leq n\} \subset \{\sigma(h'(i')) \mid 0 \leq i' \leq n'\}.$$

We have the obvious projection  $q: \mathbf{P} \rightarrow \mathbf{I}$  given by  $(\mathbf{n}, \mathbf{p}, h) \mapsto \mathbf{n}$ .

For any integer  $n \geq 1$ , one has a map  $\mathbf{n} \rightarrow \text{Fun}(\mathbf{1}, \mathbf{n})$  that carries each  $i$  to the identity map at  $i$ . This defines a section  $i_0: \mathbf{I} \rightarrow \mathbf{P}$  of  $q$ . There is another section  $i_1: \mathbf{I} \rightarrow \mathbf{P}$  of  $q$ , which carries an object  $\mathbf{n}$  to the triple  $(\mathbf{n}, \mathbf{2}, [00 \rightarrow 0n \rightarrow nn])$ .

For any integer  $N \geq 1$ , we can consider the full subcategory  $\mathbf{I}_{\leq N} \subset \mathbf{I}$  spanned by those objects  $\mathbf{n}$  such that  $n \leq N$ , and we can consider the pullback  $\mathbf{P}_{\leq N} := \mathbf{I}_{\leq N} \times_{\mathbf{I}} \mathbf{P}$ . One shows easily that when restricted to  $\mathbf{I}_N$ , there exist a zigzag of natural transformations connecting  $i_0$  and  $i_1$  and a zigzag of natural transformations connecting  $i_0 \circ q$  and the identity map on  $\mathbf{P}_{\leq N}$ .

Now for any finite simplicial set  $X$  and any map  $g: X \rightarrow NR_{[K_0 \rightarrow K_1]}$ , let  $P_g$  denote the fiber product

$$\begin{array}{ccc} P_g & \longrightarrow & NP \\ \downarrow & & \downarrow \\ X & \longrightarrow & NR_{[K_0 \rightarrow K_1]} \longrightarrow NI. \end{array}$$

The sections  $i_0$  and  $i_1$  pull back to sections of  $P_g \rightarrow X$ . Since  $X$  is finite, there exists an integer  $N \geq 1$  such that the composite  $X \rightarrow NR_{[K_0 \rightarrow K_1]} \rightarrow NI$  factors through  $NI_{\leq N}$ . Hence the zigzag of homotopies between  $i_0$  and  $i_1$  and between  $i_0 \circ q$  and the identity map on  $\mathbf{P}_{\leq N}$  lift, and we deduce that  $P_g \rightarrow X$  is an equivalence.

Now set  $P_g^* = P_g/i_1(X)$ ; let  $v$  be the vertex corresponding to  $i_1(X)$ , and regard  $X$  as a simplicial subset of  $P_g^*$  via  $i_0$ . It is thus clear that  $P_g^*$  is weakly contractible.

A.7. Now we define the extension of  $g$  to a map  $G: P_g^* \rightarrow NR_{[K_0 \rightarrow K_1]}$ . Applying the coronal factorization repeatedly, one obtains, functorially, for any object

$$[K_0 = L_0 \rightarrow L_1 \rightarrow \cdots \rightarrow L_n = K_1] \in R_{[K_0 \rightarrow K_1]},$$

a functor  $\bar{L}: \text{Fun}(\mathbf{1}, \mathbf{n}) \rightarrow \Psi \wr \Phi$  such that:

- for any integer  $0 \leq i \leq n$ , one has  $\bar{L}_{ii} = L_i$ ,
- the factorization  $\bar{L}_{ii} \rightarrow \bar{L}_{ij} \rightarrow \bar{L}_{jj}$  is a coronal factorization, and
- the factorization  $K_0 = L_0 \rightarrow L_{0n} \rightarrow L_n = K_1$  is equal to our chosen coronal factorization  $K_0 \rightarrow K_{01} \rightarrow K_1$ .

It is easy to check that the morphisms

$$\bar{L}_{ij} \longrightarrow \bar{L}_{ik}, \quad \bar{L}_{ik} \longrightarrow \bar{L}_{jk}, \quad \text{and} \quad \bar{L}_{ik} \longrightarrow \bar{L}_{(i+1)(k+1)}$$

all have rectangular fibers. This defines an extension of  $g$  to a map

$$P_g \longrightarrow NR_{[K_0 \longrightarrow K_1]},$$

which then factors through  $P_g^*$ , as desired.

#### APPENDIX B. A PROOF OF TH. 6.9

Suppose  $\Phi$  a perfect operator category. There are two unit axioms and an associativity axiom to be checked for the endofunctor  $T$  equipped with the natural transformations  $\iota$  and  $\mu$ .

B.1. The first unit axiom is the assertion that  $(T\iota) \circ \mu = \text{id}_T$ . To verify this, we first claim that for any morphism  $\phi: I \longrightarrow T$  of  $\Phi$ , one has  $\sigma_I \circ \iota_I = U\lambda_I$ , where  $\lambda: \text{id}_{(\Phi/T)} \longrightarrow E \circ \text{fib}$  is the unit natural transformation. Indeed, one has a commutative diagram

$$\begin{array}{ccccc} & & TJ & & \\ & \nearrow \iota_J & \downarrow & \searrow \sigma_J & \\ J & & TT & & T(J_t) \\ & \searrow & \downarrow \chi_t & \nearrow & \\ & & T & & \end{array}$$

and the special fiber of the composite is the identity on  $J_t$ . Hence one has

$$(T\iota) \circ \mu = (UE\kappa) \circ (U\lambda E).$$

The triangle identity for the adjunction  $(\text{fib}, E)$  now implies that

$$(T\iota) \circ \mu = U \text{id}_E = \text{id}_T,$$

as desired. The second unit axiom is analogous.

B.2. It remains to prove the associativity condition; that is, that for any object  $I \in \Phi$ , the diagram

$$(B.2.1) \quad \begin{array}{ccc} T^3 I & \xrightarrow{T\mu_I} & T^2 I \\ \mu_{TI} \downarrow & & \downarrow \mu_I \\ T^2 I & \xrightarrow{\mu_I} & TI \end{array}$$

commutes.

We begin with the following key technical lemma.

B.3. **Lemma.** *For any object  $I$  of  $\Phi$ , the rectangle*

$$\begin{array}{ccc} I & \xrightarrow{\iota_I} & TI & \xrightarrow{\iota_{TI}} & T^2 I \\ \parallel & & & & \downarrow \mu_I \\ I & \xrightarrow{\iota_I} & TI & & \end{array}$$

*is a pullback.*

*Proof.* Form the larger diagram

$$\begin{array}{ccccc}
 I & \xrightarrow{\iota_I} & TI & \xrightarrow{\iota_{TI}} & T^2I \\
 \parallel & & & & \downarrow \mu_I \\
 I & \xrightarrow{\iota_I} & TI & & \\
 \downarrow & & & & \downarrow e_I \\
 * & \xrightarrow{t} & T & & 
 \end{array}$$

The lower diagram is a pullback diagram; hence it suffices to show that the exterior rectangle is a pullback. To verify this, observe that  $\mu_* = \chi_t$ ; hence the naturality of  $\mu$  implies that the square

$$\begin{array}{ccc}
 T^2I & \xrightarrow{Te_I} & TT \\
 \mu_I \downarrow & & \downarrow \chi_t \\
 TI & \xrightarrow{e_I} & T
 \end{array}$$

commutes, since  $e_I = T(!)$ . All the rectangles of the diagram

$$\begin{array}{ccccc}
 I & \xrightarrow{\iota_I} & TI & \xrightarrow{\iota_{TI}} & T^2I \\
 \downarrow & & & & \downarrow Te_I \\
 * & \xrightarrow{t} & T & \xrightarrow{\iota_T} & TT \\
 \parallel & & & & \downarrow \chi_t \\
 * & \xrightarrow{t} & T & & T
 \end{array}$$

are pullback squares, whence the desired result. □

Now we can prove the associativity.

B.4. The first claim is that, using the structure morphism  $e_I: TI \rightarrow T$ , one may view the square (B.2.1) as a square of  $(\Phi/T)$ ; that is, we claim that the diagram

(B.4.1)

$$\begin{array}{ccccc}
 & & T^2I & \xrightarrow{\mu_I} & TI \\
 & \nearrow T\mu_I & & & \searrow e_I \\
 T^3I & & & & T \\
 & \searrow \mu_{TI} & & & \nearrow e_I \\
 & & T^2I & \xrightarrow{\mu_I} & TI
 \end{array}$$

commutes. To see this, consider the cube

$$\begin{array}{ccccc}
 & & T^2T & \xrightarrow{\mu_T} & TT \\
 & T^2e_I \nearrow & \downarrow \mu_{TI} & & \downarrow Te_I \\
 T^3I & \xrightarrow{\quad} & T^2I & \xrightarrow{\quad} & T \\
 \downarrow T\mu_I & & \downarrow T\chi_t & & \downarrow \chi_t \\
 & T^2I & \xrightarrow{\quad} & TT & \xrightarrow{\quad} & T \\
 & \downarrow Te_I & & \downarrow \mu_I & & \downarrow e_I \\
 T^2I & \xrightarrow{\quad} & TI & \xrightarrow{\quad} & T
 \end{array}$$

The outer square commutes since each composite  $T^2T \rightarrow T$  is a conservative morphism  $(T^2T, \iota_{TT}\iota_T(t)) \rightarrow (T, t)$ . The top square commutes by naturality, and the commutativity of the two side faces and the bottom face follows from the commutativity of (6.8.1). Thus the outer rectangle

$$\begin{array}{ccc}
 T^3I & \xrightarrow{\mu_{TI}} & T^2I \\
 \downarrow T^2e_I & & \downarrow Te_I \\
 T^2T & \xrightarrow{\mu_T} & TT \\
 \downarrow T\chi_t & & \downarrow \chi_t \\
 TT & \xrightarrow{\chi_t} & T
 \end{array}$$

commutes, and since the bottom square of the diagram

$$\begin{array}{ccc}
 T^3I & \xrightarrow{\mu_{TI}} & T^2I \\
 \downarrow T\mu_I & & \downarrow \mu_I \\
 T^2I & \xrightarrow{\mu_I} & TI \\
 \downarrow Te_I & & \downarrow e_I \\
 TT & \xrightarrow{\chi_t} & T
 \end{array}$$

commutes, it follows that (B.4.1) does as well.

Now write  $K$  for the special fiber of the composite

$$T^3I \xrightarrow{\mu_{TI}} T^2I \xrightarrow{\mu_I} TI \xrightarrow{e_I} T,$$

and write  $L$  for the special fiber of the composite

$$T^3I \xrightarrow{T\mu_I} T^2I \xrightarrow{\mu_I} TI \xrightarrow{e_I} T.$$

By adjunction it suffices to show that the two morphisms of special fibers  $K \rightarrow I$  and  $L \rightarrow I$  are equal. To compute  $K$ , consider the following diagram

$$\begin{array}{ccccccc}
I & \xrightarrow{\iota_I} & TI & \xrightarrow{\iota_{TI}} & T^2I & \xrightarrow{\iota_{T^2I}} & T^3I \\
\parallel & & \parallel & & & & \downarrow \mu_{TI} \\
I & \xrightarrow{\iota_I} & TI & \xrightarrow{\iota_{TI}} & T^2I & & \\
\parallel & & & & & & \downarrow \mu_I \\
I & \xrightarrow{\iota_I} & TI & & & & \\
\downarrow & & & & & & \downarrow e_I \\
* & \xrightarrow{t} & & & & & T
\end{array}$$

By B.3, every rectangle of this diagram is a pullback. To compute  $L$ , consider the following diagram

$$\begin{array}{ccccccc}
I & \xrightarrow{\iota_I} & TI & \xrightarrow{\iota_{TI}} & T^2I & \xrightarrow{\iota_{T^2I}} & T^3I \\
\parallel & & \parallel & & & & \downarrow T\mu_I \\
I & \xrightarrow{\iota_I} & TI & \xrightarrow{\iota_{TI}} & T^2I & & \\
\parallel & & & & & & \downarrow \mu_I \\
I & \xrightarrow{\iota_I} & TI & & & & \\
\downarrow & & & & & & \downarrow e_I \\
* & \xrightarrow{t} & & & & & T
\end{array}$$

Again by B.3, and since  $T$  preserves all pullbacks, every rectangle of this diagram is a pullback. Hence  $K = L = I$ , and the morphisms  $K \rightarrow I$  and  $L \rightarrow I$  are each simply the identity.

This completes the proof of the theorem.

#### APPENDIX C. A PROOF OF TH. 6.15

Fix an admissible functor  $F: \Psi \rightarrow \Phi$  between perfect operator categories. We have to show that the two diagrams (6.14.1) commute with  $C = \Psi$ ,  $D = \Phi$ , and  $\eta = \alpha_F$ .

C.1. Note that  $F$  induces a functor

$$F_{/TT}: (\Psi/T_\Psi T_\Psi) \rightarrow (\Phi/T_\Phi T_\Phi)$$

that assigns to any object  $I \rightarrow T_\Psi T_\Psi$  the composite

$$FI \rightarrow FT_\Psi T_\Psi \xrightarrow{\alpha_{F, T_\Psi}} T_\Phi FT_\Psi \xrightarrow{T_\Phi \chi_{F(\iota_\Psi)}} T_\Phi T_\Phi.$$

The commutativity of the first diagram of (6.14.1) follows directly from the following.

C.2. **Lemma.** *The following diagram of natural transformations of functors*

$$(\Phi/T_\Phi) \rightarrow (\Psi/T_\Psi)$$

commutes:

$$\begin{array}{ccc}
F/T \circ \chi_{t_\Psi, !} \circ E_{\Psi, /T_\Psi} & \xlongequal{\quad} & \chi_{t_\Phi, !} \circ F/TT \circ E_{\Psi, /T_\Psi} \xrightarrow{\chi_{t_\Phi, !} \circ \alpha_F} \chi_{t_\Phi, !} \circ E_{\Phi, /T_\Phi} \circ F/T \\
\downarrow F/T \circ \sigma_\Psi & & \downarrow \sigma_\Phi \circ F/T \\
F/T \circ E_\Psi \circ \text{fib}_\Psi & \xrightarrow{\alpha_F \circ \text{fib}_\Psi} & E_\Phi \circ F \circ \text{fib}_\Psi \xlongequal{\quad} E_\Phi \circ \text{fib}_\Phi \circ F/T.
\end{array}$$

*Proof.* Suppose  $\psi: J \rightarrow T_\Psi$  a morphism. The claim is that the diagram

$$\begin{array}{ccc}
FT_\Psi J & \longrightarrow & T_\Phi FJ \\
\downarrow & & \downarrow \\
FT_\Psi J_{t_\Psi} & \longrightarrow & T_\Phi (FJ)_{F(t_\Psi)}
\end{array}$$

of  $(\Phi/T_\Phi)$  commutes, where the structure morphisms in question are

$$\begin{array}{ccccccc}
FT_\Psi J & \xrightarrow{FT_\Psi \psi} & FT_\Psi T_\Psi & \xrightarrow{F(\chi_{t_\Psi})} & FT_\Psi & \xrightarrow{\chi_{F(t_\Psi)}} & T_\Phi, \\
T_\Phi FJ & \xrightarrow{T_\Phi F \psi} & T_\Phi FT_\Psi & \xrightarrow{T_\Phi \chi_{F(t_\Psi)}} & T_\Phi T_\Phi & \xrightarrow{\chi_{t_\Phi}} & T_\Phi, \\
FT_\Psi J_{t_\Psi} & \xrightarrow{F e_{J_{t_\Psi}}} & FT_\Psi & \xrightarrow{\chi_{F(t_\Psi)}} & T_\Phi, & & 
\end{array}$$

and of course the usual structure morphism  $e_{F(J_{t_\Psi})}: T_\Phi F(J_{t_\Psi}) \rightarrow T_\Phi$ . By adjunction it suffices to show that the square of special fibers

$$\begin{array}{ccc}
(FT_\Psi J)_{t_\Phi} & \longrightarrow & (T_\Phi FJ)_{t_\Phi} \\
\downarrow & & \downarrow \\
(FT_\Psi J_{t_\Psi})_{t_\Phi} & \longrightarrow & (T_\Phi (FJ)_{F(t_\Psi)})_{t_\Phi}
\end{array}$$

commutes. But since the special fiber of the morphisms  $\alpha_{F, J}$  and  $\alpha_{F, J_{t_\Psi}}$  are each the identity, the special fiber square is in particular commutative.  $\square$

C.3. We now prove the commutativity of the second diagram of (6.14.1). The claim is that for any object  $J$  of  $\Psi$ , the diagram

$$\begin{array}{ccc}
& & FT_\Psi J \\
F \iota_J & \nearrow & \downarrow \alpha_{F, J} \\
FJ & & T_\Phi FJ \\
\iota_{FJ} F & \searrow & 
\end{array}$$

commutes. By composing with the structure map  $e_{FJ}: T_\Phi FJ \rightarrow T_\Phi$ , this can be regarded as a diagram of  $(\Phi/T_\Phi)$ . Hence it suffices to verify that the special fiber triangle

$$\begin{array}{ccc}
& & (FT_\Psi J)_{t_\Phi} \\
(FJ)_{t_\Phi} & \nearrow & \downarrow \\
& & (T_\Phi FJ)_{t_\Phi}
\end{array}$$

commutes. But since the special fiber of  $\alpha_{F,J}: FT_{\Psi}J \rightarrow T_{\mathbb{F}}FJ$  is the identity on  $FJ$ , the special fiber triangle commutes.

This completes the proof of the theorem.

#### REFERENCES

1. D. W. Anderson, *Spectra and  $\Gamma$ -sets*, Algebraic topology (Proc. Sympos. Pure Math., Vol. XXII, Univ. Wisconsin, Madison, Wis., 1970), Amer. Math. Soc., Providence, R.I., 1971, pp. 23–30. MR 51 #4232
2. D. Ayala and R. Hepworth, *Configuration spaces and  $\theta_n$* , To appear in Proc. Amer. Math. Soc.
3. C. Barwick, *On left and right model categories and left and right Bousfield localizations*, Homology, Homotopy Appl. **12** (2010), no. 2, 245–320. MR 2771591 (2012a:18031)
4. C. Barwick and D. M. Kan, *Relative categories: another model for the homotopy theory of homotopy theories*, Indag. Math. (N.S.) **23** (2012), no. 1-2, 42–68. MR 2877401 (2012m:55019)
5. C. Barwick and C. Schommer-Pries, *On the unicity of the homotopy theory of higher categories*, Preprint [arXiv:1112.0040v1](https://arxiv.org/abs/1112.0040v1), December 2011.
6. M. A. Batanin, *The Eckmann-Hilton argument and higher operads*, Adv. Math. **217** (2008), no. 1, 334–385. MR MR2365200 (2008j:18008)
7. C. Berger, *Iterated wreath product of the simplex category and iterated loop spaces*, Adv. Math. **213** (2007), no. 1, 230–270. MR MR2331244
8. J. M. Boardman and R. M. Vogt, *Tensor products of theories, application to infinite loop spaces*, J. Pure Appl. Algebra **14** (1979), no. 2, 117–129. MR 524181 (80f:55012)
9. P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR 2003j:18018
10. A. Joyal and M. Tierney, *Quasi-categories vs Segal spaces*, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 277–326. MR 2342834 (2008k:55037)
11. T. Leinster, *Homotopy algebras for operads*, Preprint, 2000.
12. J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, no. 170, Princeton University Press, 2009, Also available from <http://www.math.harvard.edu/~lurie>.
13. ———, *Higher algebra*, Preprint from the web page of the author., February 2012.
14. J. P. May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin, 1972, Lectures Notes in Mathematics, Vol. 271. MR 0420610 (54 #8623b)
15. J. P. May and R. Thomason, *The uniqueness of infinite loop space machines*, Topology **17** (1978), no. 3, 205–224. MR MR508885 (80g:55015)
16. I. Moerdijk and I. Weiss, *Dendroidal sets*, Algebr. Geom. Topol. **7** (2007), 1441–1470. MR 2366165 (2009d:55014)
17. C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007 (electronic). MR MR1804411 (2002a:55020)
18. A. Robinson, *Gamma homology, Lie representations and  $E_{\infty}$  multiplications*, Invent. Math. **152** (2003), no. 2, 331–348. MR 1974890 (2004c:55020)
19. G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312. MR 50 #5782
20. J. D. Stasheff, *Homotopy associativity of  $H$ -spaces. I, II*, Trans. Amer. Math. Soc. **108** (1963), 275–292; *ibid.* **108** (1963), 293–312. MR 0158400 (28 #1623)
21. Ross Street, *Fibrations in bicategories*, Cahiers Topologie Géom. Différentielle **21** (1980), no. 2, 111–160. MR 574662 (81f:18028)
22. R. W. Thomason, *Uniqueness of delooping machines*, Duke Math. J. **46** (1979), no. 2, 217–252. MR 534053 (80e:55013)
23. B. Toën, *Dualité de Tannaka supérieure*, MPI Preprint MPIM2000-57, 2000.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, DEPARTMENT OF MATHEMATICS, BUILDING 2, 77 MASSACHUSETTS AVENUE, CAMBRIDGE, MA 02139-4307, USA  
*E-mail address:* [clarkbar@gmail.com](mailto:clarkbar@gmail.com)