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EULER'S GAMMA FUNCTION AND THE FIELD WITH ONE ELEMENT

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Contents

Introduction

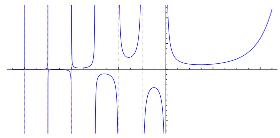
EULER'S GAMMA FUNCTION is the analytic continuation of the Mellin transform of the negative exponential function:

$$\Gamma(s) = \int_0^{+\infty} t^s \exp(-t)\, d\log t.$$

This meromorphic function has profound arithmetic significance. For instance, we have Riemann's *functional equation*: if ζ is the usual Riemann zeta function, we may set

$$\Xi(s) \coloneqq (2\pi^s)^{-1/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Then one has $\Xi(1-s) = \Xi(s)$. So, in effect, the Gamma factor plays the role of Euler factor at the infinite place.



The Gamma function has no zeroes, and its only poles are simple poles at nonpositive integers.

The Riemann zeta function is the analytic continuation of $\sum_{n\geq 1} n^{-s}$.

OUR GUIDING QUESTION is imprecise, but tantalising. Why should Γ carry this arithmetic significance? Why do factors that involve this ostensibly innocuous meromorphic function – whose definition lies wholly in the domain of classical analysis – serve as a means to *complete* the analytic functions of number theory? What about the Γ function permits one to employ it to define the local factor of ζ functions (and, more generally, L functions) at infinite places?

The purpose of this course is to attempt to answer this question by connecting this humble function with a largely hypothetical object called the *field with one element*, $\mathbf{F}_1$. There is, of course, no field with only one element, but this object is nevertheless meant to serve as a kind of kōan whose contemplation leads to some appreciation of the analogous behaviour of function fields over finite fields and number fields. In effect, $\mathbf{F}_1$ is to be conceived of as a finite field, and number rings are to be conceived of the rings of functions on affine $\mathbf{F}_1$-varieties, and compactifying these varieties over $\mathbf{F}_1$ is to be interpreted as the addition of some infinite primes. Our aim is to offer some attempt at a description of a *precise* form algebraic geometry over $\mathbf{F}_1$ – particularly as proposed in the distinct but related approaches of James

Borger and Alain Connes – that is sufficient to account for the Γ factors corresponding to the infinite places.

We will begin with a purely analytic discussion of the basic properties of the Gamma function and the Mellin transform; in particular, we will encounter a number of interesting functions, and we will prove Ramanujan's Master Theorem. This much should be understandable to anyone with a basic understanding of real and complex analysis. Our first effort to develop a conceptual explanation of the Gamma factors will then be Tate's proof of the functional equations for Dedekind zeta functions and Hecke L functions. For this, some exposure to class field theory would be helpful. Deninger described Serre's Gamma factors of motivic L functions in terms of regularized determinants and an arithmetic cohomology theory; we will explain his results, and we will enrich his description by passing to Connes and Consani's description in terms of cyclic homology. This last clump of material is rather more involved, and we will be inexorably lead into deeper bits of algebra.

PLEASE NOTE THAT this text contains many incomplete or sketched proofs. This is deliberate: the reader is asked to supply these proofs as a means of coming to grips with the ideas herein.

1

The analytic picture

1.1 The Mellin transform

To define Euler's Γ function, we will employ an analytic tool – the *multiplicative Fourier transform*, or, as it is more traditionally called, the *Mellin transform*. In effect, the Mellin transform acts as the Fourier transform on the multiplicative topological group $\mathbf{R}_{>0}$ relative to the multiplicative Haar measure $d\log t$.

1.1.1 Notation. We will use the *Landau symbols*. Suppose a a point of some topological space, suppose $g\colon V\setminus a \longrightarrow \mathbf{R}_{>0}$ and $f\colon V\setminus a \longrightarrow \mathbf{C}$ functions defined in a punctured neighborhood $V\setminus a$. Then we will write

$$f(x) = O(g(x)) \text{ as } x \longrightarrow a$$

if and only if for some neighborhood U of a and some $C > 0$, one has $|f(x)| \leq Cg(x)$ for every $x \in U$. If the implicit constant C is liable to depend upon an auxiliary parameter t, then one writes $f(x) = O_t(g(x))$ as $x \longrightarrow a$.

One particular instance of this is the following. A sequence $(\phi_m)_{m\in\mathbf{N}}$ of functions on $V\setminus a$ provide an *asymptotic expansion*

$$f(x) \sim \sum_{m\in\mathbf{N}} a_m\phi_m(x)$$

of f near a if and only if, for any $N \in \mathbf{N}$, one has

$$f(x) = \sum_{m=1}^{N-1} a_m\phi_m(x) + O_N(\phi_N(x)) \text{ as } x \longrightarrow a.$$

Example. One has $\exp(x) = 1 + x + x^2/2 + O(x^3)$ as $x \longrightarrow 0$.

Example. A function f on $\mathbf{R}_{>0}$ is bounded if and only if $f(x) = O(1)$ as $x \longrightarrow +\infty$.

Example. If f is analytic near the origin, one has $f(z) = a_0 + a_1z + a_2z^2 + \cdots + a_Nz^N + O_N(|z|^{N+1})$ as $z \longrightarrow 0$, where $a_0, a_1, \ldots, a_N$ are the first N Taylor coefficients. Hence Taylor expansions are a kind of asymptotic expansion.

Observe that no assumption about the convergence of the series $\sum_{m\in\mathbf{N}} a_m\phi_m(x)$ is made.

We also write

$$f(x) = o(g(x)) \text{ as } x \longrightarrow a$$

if and only if $f(x)/g(x) \longrightarrow 0$ as $x \longrightarrow a$.

We follow the analytic number theorists and write

$$f(x) = \Omega(g(x)) \text{ as } x \longrightarrow a$$

for the negation of the assertion $f(x) = o(g(x))$ as $x \longrightarrow a$; that is, $f(x) = \Omega(g(x))$ as $x \longrightarrow a$ if and only if

$$\limsup_{x \longrightarrow a} \frac{|f(x)|}{g(x)} > 0.$$

In more detail, we write

$$f(x) = \Omega_+(g(x)) \text{ as } x \longrightarrow a$$

if and only if

$$\limsup_{x \longrightarrow a} \frac{f(x)}{g(x)} > 0,$$

we write

$$f(x) = \Omega_-(g(x)) \text{ as } x \longrightarrow a$$

if and only if

$$\limsup_{x \longrightarrow a} \frac{f(x)}{g(x)} < 0,$$

and we write

$$f(x) = \Omega_\pm(g(x)) \text{ as } x \longrightarrow a$$

if and only if both $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ as $x \longrightarrow a$.

Let us also employ the following notation for strips in the complex plane. When $\alpha, \beta \in \mathbf{R} \cup \{-\infty, +\infty\}$, we write

$$\begin{aligned} \langle \alpha, \beta \rangle &:= \{s \in \mathbf{C} \mid \Re(s) \in [\alpha, \beta]\}; \\ \rangle \alpha, \beta \langle &:= \{s \in \mathbf{C} \mid \Re(s) \in]\alpha, \beta[\}. \end{aligned}$$

We may also have occasion to use the rather odd-looking notations $\langle \alpha, \beta \langle$ and $\rangle \alpha, \beta \rangle$ as well.

Example. One has $\log x = o(x)$ as $x \longrightarrow +\infty$.

Example. One has $7x = o(x^2)$ as $x \longrightarrow +\infty$.

Example. As a function on $\mathbf{R}$, one has $\sin x = \Omega(1)$ as $x \longrightarrow +\infty$, and even $\sin x = \Omega_\pm(1)$ as $x \longrightarrow +\infty$.

Example. As a function on $\mathbf{R}$, one has $1 + \sin x = \Omega_+(1)$ as $x \longrightarrow +\infty$, but $1 + \sin x \neq \Omega_-(1)$ as $x \longrightarrow +\infty$.

1.1.2 Definition. Suppose $f : \mathbf{R}_{>0} \longrightarrow \mathbf{C}$ a function that is absolutely integrable on $[0, r]$ for any $r \in \mathbf{R}_{>0}$, and assume that

$$f(t) = O(t^{-\alpha}) \text{ as } t \to 0,\ t > 0 \quad \text{and} \quad f(t) = O(t^{-\beta}) \text{ as } t \longrightarrow +\infty.$$

Then the integral

$$M\{f\}(s) := \int_{\mathbf{R}_{>0}} t^s f(t)\, d\log t$$

converges on the strip $\rangle\alpha, \beta\langle$ and it defines a holomorphic function there. We call the holomorphic function $M\{f\}$ the *Mellin transform of* f, and we call $\rangle\alpha, \beta\langle$ the *strip of definition*.

Note that $|M\{f\}(s)|$ is dominated by the sum

$$\int_0^1 |f(t)| t^{\mathbb{R}(s)-1}\, dt + \int_1^{+\infty} |f(t)| t^{\mathbb{R}(s)-1}\, dt,$$

and this in turn is dominated by

$$C \int_0^1 t^{\mathbb{R}(s)-\alpha-1}\, dt + C \int_1^{+\infty} t^{\mathbb{R}(s)-\beta-1}\, dt.$$

The first summand converges for $\mathbb{R}(s) > \alpha$, and the second converges for $\mathbb{R}(s) < \beta$.

Here is a list of examples of Mellin transforms. The verifications are left to the motivated reader.

1.1.3 Example. *If* $a > 0$, *then consider the ray* $[a, +\infty[$ *and its characteristic function* $\chi_{[a,+\infty[}$. *Then the Mellin transform of* $\chi_{[a,+\infty[}$ *is given by*

$$M\left\{\chi_{[a,+\infty[}\right\}(s) = -\frac{a^s}{s}$$

with strip of definition $s \in \rangle -\infty, -a\langle$.

1.1.4 Example. *No polynomial admits a Mellin transform, because the integral never converges.*

1.1.5 Example. *The characteristic function of the open interval* $]0, 1[$ *admits a Mellin transform; it is given by*

$$M\left\{\chi_{]0,1[}\right\}(s) = \frac{1}{s}$$

with strip of definition $s \in \rangle 0, +\infty\langle$.

1.1.6 Example. *The Mellin transform of the function* $f(x) = (1 + x)^{-1}$, *is given by*

$$M\{f\}(s) = \pi \csc(\pi s)$$

with strip of definition $s \in \rangle 0, 1\langle$.

1.1.7 Example. *The Mellin transform of the function* $f(x) = (1 - x)^{-1}$ *is given by*

$$M\{f\}(s) = \pi \cot(\pi s)$$

with strip of definition $s \in \rangle 0, 1\langle$.

1.1.8 Example. *The Mellin transform of the function $f(x) = \tan^{-1}(x)$ is given by*

$$M\{f\}(s) = -\frac{\pi}{2}s^{-1}\sec\left(\frac{\pi}{2}s\right)$$

with strip of definition $s \in \rangle -1, 0\langle$.

1.1.9 Example. *The Mellin transform of the function $f(x) = \cot^{-1}(x)$ is given by*

$$M\{f\}(s) = \frac{\pi}{2}s^{-1}\sec\left(\frac{\pi}{2}s\right)$$

with strip of definition $s \in \rangle 0, 1\langle$.

1.1.10 Example. *The Mellin transform of the function*

$$f(x) = \log\left|\frac{1+x}{1-x}\right|,$$

is given by

$$M\{f\}(s) = \pi s^{-1}\tan\left(\frac{\pi}{2}s\right).$$

with strip of definition $s \in \rangle -1, 1\langle$.

THE MELLIN TRANSFORM enjoys some basic compatibilites with operations on functions that we explore now. Of course it goes without saying that the Mellin transform is linear, but there are even better compatibilities worth studying.

1.1.11 Lemma. *Suppose*

$$f: \mathbf{R}_{>0} \longrightarrow \mathbf{C}$$

a function with Mellin transform $M\{f\}$.

These identities hold in the appropriate strip of definition. The proof of these claims – as well as the task of finding the appropriate strip of definition – is left to the reader.

(i) If $a > 0$ and $g(x) = f(ax)$, then $M\{g\}(s) = a^{-s}M\{f\}(s)$.

(ii) If $z \in \mathbf{C}$ and $g(x) = x^z f(x)$, then $M\{g\}(s) = M\{f\}(z+s)$.

(iii) If $a > 0$ and $g(x) = f(x^a)$, then $M\{g\}(s) = a^{-1}M\{f\}(a^{-1}s)$.

(iv) If $a < 0$ and $g(x) = f(x^a)$, then $M\{g\}(s) = -a^{-1}M\{f\}(a^{-1}s)$.

(v) If $g(x) = f'(x)$, then $M\{g\}(s) = (1-s)M\{f\}(s-1)$.

(vi) If $g(x) = \log(x)f(x)$, then $M\{g\} = \frac{d}{ds}M\{f\}(s)$.

1.1.12. Consider the topological group isomorphism $\exp\colon \mathbf{R} \longrightarrow \mathbf{R}_{>0}$; if $f\colon \mathbf{R}_{>0} \longrightarrow \mathbf{C}$, then it is easy to see that

$$M\{f\}(-2\pi i y) = F\{f \circ \exp\}(y),$$

where F is the Fourier transform.

Here of course $\mathbf{R}$ is considered additively and $\mathbf{R}_{>0}$ is considered multiplicatively.

1.1.13 Definition. If ϕ is a holomorphic function on a strip $\rangle\alpha, \beta\langle$, then for any $x \in \mathbf{R}_{>0}$ and for any $c \in \,]\alpha, \beta[$, the formula

$$M^{-1}\{\phi\}(x) \coloneqq \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \phi(s)x^{-s}\, ds,$$

defines a function $M^{-1}\{\phi\}\colon \mathbf{R}_{>0} \longrightarrow \mathbf{C}$, called the *inverse Mellin transform* of ϕ.

This contour integral is independent of c thanks to the Cauchy integral formula.

1.1.14 Theorem (Mellin inversion)**.** *Suppose $f\colon \mathbf{R}_{>0} \longrightarrow \mathbf{C}$ a function with Mellin transform $M\{f\}$. Then one has*

$$f = M^{-1}\{M\{f\}\}.$$

We omit this proof.

A COMMON SITUATION is that the Mellin transform of a function admits an analytic continuation to a meromorphic function on the complex plane.

1.1.15 Notation. Let us write

$$-\mathbf{N} \coloneqq \{-1, -2, \dots\}, \text{ and } -\mathbf{N}_0 \coloneqq \{0, -1, \dots\}.$$

1.1.16 Proposition. *Suppose $f\colon \mathbf{R}_{>0} \longrightarrow \mathbf{C}$ an infinitely differentiable function of rapid decay at infinity, and assume that we have an asymptotic expansion*

$$f(x) \sim \sum_{m \in \mathbf{N}_0} a_m x^m \text{ as } x \longrightarrow 0.$$

Then $M\{f\}$ admits a meromorphic continuation to the complex plane with simple poles at $s = -n \in -\mathbf{N}_0$ of residue

$$\operatorname{Res}_{-n} M\{f\} = a_n.$$

By *rapid decay*, we mean that

$$\sup_{x \in \mathbf{R}_{>0}} |x^m f^{(n)}(x)| < +\infty$$

for every $m, n \geq 0$.

Proof. One writes

$$M\{f\}(s) \coloneqq \int_{\mathbf{R}_{>0}} t^s f(t)\, d\log t = \int_0^1 t^s f(t)\, d\log t + \int_1^{+\infty} t^s f(t)\, d\log t,$$

and one sees readily that the second summand is entire. The first summand can be rewritten, for any $N \in \mathbf{N}_0$, as

$$\int_0^1 t^s f(t)\, d\log t = \int_0^1 t^s \left(f(t) - \sum_{m=0}^{N-1} a_m t^m \right) d\log t + \sum_{m=0}^{N-1} \frac{a_m}{m+s}.$$

Here the first summand converges on the half-plane $\rangle -N, +\infty \langle$. This integral is thus meromorphic on $\rangle -N, +\infty\langle$ with simple poles of residue a_m at $s = -m \in \{0, -1, \ldots, -N\}$. Since N is arbitrary, we conclude. □

The same argument provides us with the following more general statement:

1.1.17 Proposition. *Suppose $f \colon \mathbf{R}_{>0} \longrightarrow \mathbf{C}$ an infinitely differentiable function of rapid decay at infinity, and assume also that there is a sequence $(\alpha_m)_{m \in \mathbf{N}}$ of complex numbers such that*

$$\lim_{m \longrightarrow +\infty} \Re(\alpha_m) = +\infty,$$

and one has an asymptotic expansion

$$f(x) \sim \sum_{m \in \mathbf{N}} a_m x^{\alpha_m} \text{ as } x \longrightarrow 0.$$

Then $M\{f\}$ admits a meromorphic continuation to the complex plane with simple poles at $s = -\alpha_n$ $(n \in \mathbf{N})$ of residue

$$\operatorname{Res}_{-\alpha_n} M\{f\} = a_n.$$

1.2 *Euler's Gamma function*

We are now prepared to introduce Euler's function Γ.

1.2.1 Definition. If $f(x) = \exp(-x)$, then we define the Gamma function as the Mellin transform of f:

$$\Gamma(s) \coloneqq M\{f\}(s) = \int_{\mathbf{R}_{>0}} t^s \exp(-t)\, d\log t$$

for $s \in \rangle 0, +\infty\langle$.

1.2.2 Theorem. *The function Γ admits an analytic continuation to a meromorphic function on $\mathbf{C}$ with simple poles at $s = -n \in -\mathbf{N}_0$ of residue*

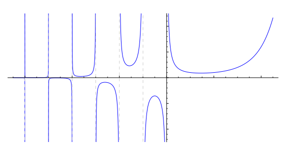


Figure 1.1: A graph of $\Gamma(s)$, for $s \in \mathbf{R}$.

$$\operatorname{Res}_{-n} \Gamma = \frac{(-1)^n}{n!}.$$

1.2.3 Example. *One has*

$$\Gamma(1) = \int_{\mathbf{R}_{>0}} \exp(-t)\, dt = 1.$$

1.2.4 Proposition (Functional Equation I)**.** *For any $s > 0$, one has*

$$\Gamma(1+s) = s\Gamma(s),$$

and, consequently,

$$\Gamma(n+s) = \Gamma(s) \prod_{k=0}^{n-1} (k+s)$$

Proof. This is immediate from Lm. 1.1.11(ii). □

1.2.5 Example. *If $n \geq 0$ is an integer, one has*

$$\Gamma(1+n) = n!$$

Gauss preferred the normalization $\Pi(s) = \Gamma(1+s)$, so that $\Pi(n) = n!$.

1.2.6. One could instead have used Lm. 1.2.4 to analyticially continue Γ from $\rangle 0, +\infty\langle$ strip-by-strip to $\mathbf{C} \setminus -\mathbf{N}_0$.

1.2.7 Definition. For $u, v \in \rangle 0, +\infty\langle$, write

$$f_v(x) = \chi_{]0,1[}(x)(1-x)^{v-1};$$

one defines the *Beta function* as

$$B(u,v) \coloneqq M\{f_v\}(u) = \int_0^1 x^{u-1}(1-x)^{v-1}\, dx.$$

1.2.8 Proposition. *For $u, v \in \rangle 0, +\infty\langle$, one has*

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

Proof. One has

$$\Gamma(u)\Gamma(v) = \iint_{\mathbf{R}_{>0} \times \mathbf{R}_{>0}} \exp(-s+t) s^{u-1} t^{v-1}\, du\, dv,$$

and the change of variables $u = xy$ and $v = x(1-y)$ can then be performed to obtain

$$\begin{aligned} \Gamma(u)\Gamma(v) &= \int_0^{+\infty} \exp(-x) x^{u+v}\, d\log x \cdot \int_0^1 y^{v-1}(1-y)^{u-1}\, dy \\ &= \Gamma(u+v)B(u,v), \end{aligned}$$

as desired. □

1.2.9 Proposition (Functional Equation II). *If $s \in \mathbf{C} \setminus \mathbf{Z}$, then one has*

$$\Gamma(s)\Gamma(1-s) = B(s, 1-s) = \pi \csc(\pi s).$$

Proof. The first equality is the previous result. It is clear that $\Gamma(1+s)\Gamma(-s) = -\Gamma(s)\Gamma(1-s)$ and $\pi\csc(\pi(1+s)) = -\pi\csc(\pi s)$, so it suffices to verify the claim for $s \in \rangle 0, 1\langle$. In this strip, if $f(x) = (1+x)^{-1}$, then we have seen that

$$M\{f\}(s) = \pi \csc(\pi s),$$

but on the other hand,

$$M\{f\}(s) = \int_0^1 x^{s-1}(1-x)^{-s}\, dx = B(s, 1-s),$$

as desired. □

1.2.10 Example. *We obtain computations directly from this:*

This offers an inefficient proof that $\int_{\mathbf{R}} \exp(-t^2)\, dt = \sqrt{\pi}$.

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

and, more generally,

The value of the Gamma function at half-integers is of particular import, since it appears in the formula for the volume of an n-ball, which we will discuss below.

$$\Gamma\left(m + \frac{1}{2}\right) = \frac{(2m-1)!!}{2^m}\sqrt{\pi},$$

where the semifactorial is defined by the formula

$$k!! = \prod_{j=0}^{\lceil k/2 \rceil - 1} (k - 2j),$$

so that

1.2.11 Definition. For any positive integer m and any $s \in \mathbf{C} \setminus -\mathbf{N}_0$, define

$$\Gamma_m(s) := \frac{m^s m!}{s(1+s)\cdots(m+s)} = \frac{m^s}{s\left(1+\frac{s}{1}\right)\cdots\left(1+\frac{s}{m}\right)}.$$

1.2.12 Theorem (Euler Product Formula). *For any* $s \in \mathbf{C} \setminus -\mathbf{N}_0$, *we have*

$$\Gamma(s) = \lim_{m \longrightarrow +\infty} \Gamma_m(s).$$

Proof. It is easy to see that $\Gamma_m(1+s) = s\left(\frac{m-1}{m}\right)^{1+s} \Gamma_{1+m}(s)$; hence the limit on the right satisfies the functional equation for Γ. We are thus reduced to proving the claim for s lying in the half plane $\rangle 0, +\infty\langle$. One has

$$\exp(-t) = \lim_{m \longrightarrow +\infty} \left(1 - \frac{t}{m}\right)^m,$$

so the absolute convergence of both the integral and the limit permits us to write

$$\Gamma(s) = \lim_{m \longrightarrow +\infty} \int_0^m \left(1 - \frac{t}{m}\right)^m t^s \, d\log t$$

for $s \in \rangle 0, +\infty\langle$. Integrating by parts, we obtain

$$\int_0^m \left(1 - \frac{t}{m}\right)^m t^s \, d\log t = \frac{m!}{s(s+1)\cdots(s+m-1)m^m} \int_0^m t^{s+m-1} \, dt = \Gamma_m(s),$$

as desired. □

1.2.12.1 Corollary. *In particular, for any* $s \in \mathbf{C} \setminus -\mathbf{N}_0$, *we have*

$$\Gamma(s) = \frac{1}{s} \prod_{n \in \mathbf{N}} \frac{\left(1 + \frac{1}{n}\right)^s}{1 + \frac{s}{n}}.$$

1.2.13 Notation. For $m \in \mathbf{N}_0$, we write h_m for the m-th *harmonic number*

$$h_m = \sum_{k=1}^m \frac{1}{k}.$$

We let γ denote the *Euler–Mascheroni constant*, defined by

$$h_N = \gamma + \log N + O\left(\frac{1}{N}\right)$$

for any positive integer N.

1.2.14 Theorem (Weierstraß Product Formula). *For any element* $s \in \mathbf{C} \setminus -\mathbf{N}_0$, *we have*

$$\Gamma(s) = \frac{\exp(-\gamma s)}{s} \prod_{n \in \mathbf{N}} \frac{\exp\left(\frac{s}{n}\right)}{1 + \frac{s}{n}}.$$

Proof. One has

$$m^s = \exp(s\log m) = \exp\left(s\log m - s\sum_{n=1}^{m}\frac{1}{n}\right)\cdot\exp\left(s\sum_{n=1}^{m}\frac{1}{n}\right),$$

whence

$$\Gamma_m(s) = \frac{1}{s}\exp\left(s\log m - s\sum_{n=1}^{m}\frac{1}{n}\right)\prod_{n=1}^{m}\frac{\exp\left(\frac{s}{n}\right)}{1+\frac{s}{n}}.$$

Letting $m \longrightarrow +\infty$, one obtains the Weierstraß Product Formula. □

1.2.15 Proposition (Gauß Product Formula)**.** *One has*

$$\prod_{k=0}^{m-1}\Gamma\left(s+\frac{k}{m}\right) = (2\pi)^{(m-1)/2}m^{1/2-ms}\Gamma(ms).$$

1.2.15.1 Corollary.

$$\prod_{k=1}^{m-1}\Gamma\left(\frac{k}{m}\right) = \frac{(2\pi)^{(m-1)/2}}{\sqrt{m}}.$$

THE PSI FUNCTION is the logarithmic derivative of the Gamma function.

1.2.16 Definition. One defines, for $s \in \mathbf{C}\setminus -\mathbf{N}_0$

$$\Psi(s) := \frac{d}{ds}\log\Gamma(s) = \frac{\Gamma'(s)}{\Gamma(s)}.$$

As a consequence of the Euler Product Formula, we deduce the following.

1.2.17 Proposition. *One has, for any* $s \in \mathbf{C}\setminus -\mathbf{N}_0$,

$$\Psi(s) = \lim_{m\longrightarrow +\infty}\left(\log m - \sum_{k=0}^{m}\frac{1}{k+s}\right)$$

On the other hand, the Weierstraß Product Formula implies the following.

1.2.18 Proposition. *One has, for any* $s \in \mathbf{C}\setminus -\mathbf{N}_0$,

$$\Psi(s) = -\gamma + \sum_{n\in\mathbf{N}}\frac{s-1}{n(s-1+n)}.$$

1.2.19 Example. *It is obvious that* $\Psi(1) = -\gamma$. *More generally,*

$$\Psi(1+m) = h_m - \gamma.$$

1.2.20 Proposition (Functional Equation I)**.** *For any* $s \in \mathbf{C} \setminus -\mathbf{N}_0$, *one has*

$$\Psi(1+s) = \frac{1}{s} + \Psi(s).$$

1.2.21 Proposition (Functional Equation II)**.** *For any* $s \in \mathbf{C} \setminus \mathbf{Z}$, *one has*

$$\Psi(s) - \Psi(1-s) = -\pi \cot(\pi s).$$

LET v_n BE THE VOLUME of the unit n-ball. Then one has

$$\begin{aligned} \pi^{n/2} &= \left(\int_{\mathbf{R}} \exp(-x^2)\,dx\right)^n \\ &= \int_{\mathbf{R}^n} \exp\left(-x_1^2 - \cdots - x_n^2\right)\,dx_1 \cdots dx_n \\ &= \int_0^1 v_n\,(-\log t)^{n/2}\,dt \\ &= v_n \int_0^{+\infty} s^{n/2} \exp(-s)\,ds \\ &= v_n \Gamma\left(1 + \frac{n}{2}\right). \end{aligned}$$

In the third equality, we're simply using the fact that

$$\int_X f\,d\mu = \int_0^{+\infty} \mu\{x \in X \mid f(x) > t\}\,dt$$

for a positive function f on a measure space (X, μ).

1.3 *Ramanujan's Master Theorem*

1.3.1 Definition. Suppose $A, P, \delta \in \mathbf{R}$ are real numbers such that $A < \pi$ and $0 < \delta \leq 1$. The *Hardy class* $H(A, P, \delta)$ consists of holomorphic functions ϕ on the half-plane $\rangle -\delta, +\infty\langle$ such that

$$\phi(s) = O(\exp(-P\Re s + A|\Im s|)).$$

1.3.2 Theorem (Ramanujan Master)**.** *Suppose* $\phi \in H(A, P, \delta)$. *Then the power series*

$$f(t) = \sum_{m \in \mathbf{N}_0} (-t)^m \phi(t)$$

converges for $t \in\,]0, \exp(P)[$ *and defines a real analytic functon* f *there. The function* f *extends to an analytic function on* $]0, +\infty[$, *and for any* $s \in\, \rangle 0, \delta\langle$, *we have*

$$M\{f\}(s) = \pi \csc(\pi s) \phi(-s).$$

Proof. Write

$$\Phi(t) := \sum_{m\in\mathbf{N}_0} (-t)^m \phi(m);$$

the growth conditions given ensure that $\Phi(t)$ converges for $t \in]0, \exp(P)[$. The Cauchy Residue Theorem gives

$$\Phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \pi\csc(\pi s)\phi(-s)t^{-s}\,ds$$

for any $c \in]0, \delta[$. The integral on the right converges uniformly on compact subsets of $]0, +\infty[$, and now the claim now follows from the Mellin inversion formula. □

Writing $\lambda(s) = \phi(s)\Gamma(1+s)$, we obtain the formula

$$\Gamma(s)\lambda(-s) = \int_0^{+\infty} t^s \sum_{m\in\mathbf{N}_0} \frac{(-t)^m}{m!}\lambda(m)\,d\log t.$$

Much about this formula is interesting, but one way to think of it is to say that the Mellin transform of a power series interpolates the coefficients of that power series.

1.3.3 Example. *Assume that f admits an expansion of the form*

$$f(t) = \sum_{m\in\mathbf{N}_0} \frac{\lambda(m)}{m!}(-t)^m.$$

Then one has

$$M\{f\}(s) = \Gamma(s)\lambda(-s).$$

1.3.4 Example. *Let's write*

$$f(t) := \frac{1}{\exp(t)-1} = \sum_{m\in\mathbf{N}_0} \frac{B_m}{m!}t^{m-1}$$

for $t > 0$, where B_m is the m-th Bernoulli number, given by

$$B_m = \sum_{k=0}^{m} \frac{1}{1+k}\sum_{n=0}^{k}(-1)^n\binom{k}{n}n^m.$$

Clearly f is of rapid decay at infinity, and so one knows that the Mellin transform $M\{f\}$ is meromorphic on $\mathbf{C}$ with simple poles at $s = 1-m$ ($m \in \mathbf{N}_0$) of residue

$$\mathrm{Res}_{1-m} M\{f\} = \frac{B_m}{m!}.$$

Since $\exp(t) > 1$ *if* $t > 0$, *we can write* $f(t)$ *as a geometric series*

$$f(t) = \sum_{m \in \mathbf{N}} \exp(-mt),$$

so when we apply our rules for the Mellin transform, we obtain

$$M\{f\}(s) = \sum_{m \in \mathbf{N}} \Gamma(s) m^{-s} = \Gamma(s)\zeta(s),$$

where

$$\zeta(s) = \sum_{m \in \mathbf{N}} m^{-s}.$$

Since we know that Γ *is nonvanishing and meromorphic on the plane with simple poles of residue* $(-1)^m/m!$ *at* $s = -m \in -\mathbf{N}_0$, *it follows that* $\zeta(s)$ *admits a meromorphic continuation to the plane with a unique simple pole of residue* 1 *at* $s = 1$. *Moreover, the Ramanujan Master Theorem gives, for any* $-m \in -\mathbf{N}_0$,

$$\zeta(-m) = (-1)^m \frac{B_{m+1}}{m+1}$$

1.3.5 Example. *We can generalize the previous example in the following manner. The Hurwitz zeta function*

$$\zeta(s, q) = \sum_{m \in \mathbf{N}} (m+q)^{-s}.$$

Then if

$$f(t) = \frac{\exp(-qt)}{1 - \exp(-t)} - \frac{1}{t},$$

one has

$$M\{f\}(s) = \Gamma(s)\zeta(s, q).$$

On the other hand, if $B_m(q)$ *denotes the m-th Bernoulli polynomial*

$$B_m(q) = \sum_{k=0}^{m} \binom{m}{k} B_m q^{m-k},$$

then near the origin, one has

$$\frac{t \exp(qt)}{\exp(t) - 1} = \sum_{m \in \mathbf{N}_0} \frac{B_m(q)}{m!} t^m,$$

whence for $-m \in -\mathbf{N}$,

$$\zeta(1 - m, q) = -\frac{B_m(q)}{m}.$$

1.3.6 Definition. A *Dirichlet series* is a series

$$L(s) = \sum_{m \in \mathbf{N}} a_m m^{-s};$$

a *generalized Dirichlet series* is a series

$$L(s) = \sum_{m \in \mathbf{N}} a_m \lambda_m^{-s}$$

for some increasing sequence (λ_m) with $\lambda_m \longrightarrow +\infty$ that grows at least as quickly as (m^j) with $j > 0$.

1.3.7 Theorem. *Any generalized Dirichlet series $L(s) = \sum_{m \in \mathbf{N}} a_m \exp(-\lambda_m s)$ admits an abscissa of convergence σ_c with the property that $L(s)$ converges if $s \in \rangle \sigma_c, +\infty \langle$, and $L(s)$ does not converge if $s \in \rangle -\infty, \sigma_c \langle$. Furthermore, if $s_0 \in \rangle \sigma_c, +\infty \langle$, then there is a neighbourhood of s_0 on which $L(s)$ converges uniformly.*

1.3.8 Example. *Suppose*

$$L(s) = \sum_{m \in \mathbf{N}} a_m \lambda_m^{-s}$$

a generalized Dirichlet series. Now set, for $t > 0$,

$$f(t) := \sum_{m \in \mathbf{N}} a_m \exp(-\lambda_m t).$$

Assume that ϕ has an asymptotic expansion

$$f(t) \sim \sum_{n=-1}^{+\infty} b_m t^m \text{ as } t \longrightarrow 0.$$

Then

$$M\{f\}(s) = \Gamma(s) L(s),$$

whence $L(s)$ admits a meromorphic continuation to the complex plane with only one simple pole at $s = 1$, and for any $-m \in -\mathbf{N}_0$,

$$L(-m) = (-1)^m m! a_m.$$

1.3.9 Example. *A character*

$$\chi \colon (\mathbf{Z}/k)^\times \longrightarrow \mathbf{C}^\times,$$

extends to $\mathbf{Z}/k$ *by declaring* $\chi(j) = 0$ *if* j *is not a unit and then to* $\mathbf{Z}$ *via the obvious quotient map. The resulting map*

$$\chi\colon \mathbf{Z} \longrightarrow \mathbf{C}$$

is called a Dirichlet character.

One defines the Dirichlet L-series

$$L(s,\chi) = \sum_{m\in\mathbf{N}} \chi(m)m^{-s},$$

and one can follow the recipe above to write

$$f(t,\chi) = \sum_{m\in\mathbf{N}} \chi(m)\exp(-mt).$$

Then

$$M\{f(\bullet,\chi)\}(s) = \Gamma(s)L(s,\chi),$$

whence $L(s,\chi)$ *admits a meromorphic continuation to the complex plane with only one simple pole at* $s = 1$*, and for any* $-m \in -\mathbf{N}$*,*

$$L(1-m,\chi) = -\frac{B_{m,\chi}}{m},$$

where $B_{m,\chi}$ *is the generalized Bernoulli number, with*

$$\sum_{m\in\mathbf{N}} \chi(m)\frac{t\exp(mt)}{\exp(mt)-1} = \sum_{m\in\mathbf{N}_0} \frac{B_{m,\chi}}{m!}t^m.$$

1.3.10 Example. *Consider the Dirichlet series*

$$L(s) = L(s,\chi_4) = \sum_{m\in\mathbf{N}} (-1)^m(2m-1)^{-s}.$$

Then one has

$$f(t) = \frac{1}{2}\operatorname{sech} t = \frac{1}{2}\sum_{m\in\mathbf{N}_0} \frac{E_m}{m!}t^m,$$

where E_m *is the m-th Euler number. We thus obtain for any* $-m \in -\mathbf{N}_0$*,*

$$L(-m) = \frac{E_m}{2}.$$

1.3.11 Theorem. *Suppose* $L(s) = \sum_{m\in\mathbf{N}} a_m \exp(-\lambda_m s)$*. Let*

$$A(t) = \sum_{m\leq t} a_m.$$

Then for any $s \in \rangle\max\{0,\sigma_c\},+\infty\langle$*, one has*

$$M\{F\}(s) = \frac{L(s)}{s},$$

where $F(t) = A(1/t)$*.*

1.4 Theta series

1.4.1 Definition. Write

$$\theta(z) := \sum_{n\in\mathbf{Z}} \exp(i\pi n^2 z) = 1 + 2\sum_{n\in\mathbf{N}} \exp(i\pi n^2 z).$$

This series converges in the upper half-plane $\mathbf{H} = \{z \in \mathbf{C} \mid \Im(z) > 0\}$, and it defines an analytic function there, called *Jacobi's θ function.*

Let us contemplate a suitably normalized version of the Mellin transform of the function $f(x) = \frac{1}{2}(\theta(ix) - 1)$: set

$$Z(s) := M\{f\}(s/2).$$

One sees easily that the Mellin transform of $g_n(x) = \exp(-\pi n^2 x)$ can be computed as

$$M\{g_n\}(s) = \pi^{-s}\Gamma(s)n^{-2s},$$

and absolute convergence permits us to obtain, for $s \in \rangle 0, +\infty\langle$,

$$M\left\{\sum_{n\in\mathbf{N}} g_n\right\}(s) = \pi^{-s}\Gamma(s)\zeta(2s).$$

But of course $f = \sum_{n\in\mathbf{N}} g_n$, and so we conclude

1.4.2 Proposition. *One has*

$$Z(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

1.4.3 Theorem (Poisson Summation)**.** *For any (complex-valued) Schwartz function ϕ on $\mathbf{R}$, one has*

$$\sum_{n\in\mathbf{Z}} \phi(m) = \sum_{n\in\mathbf{Z}} \widehat{\phi}(m).$$

1.4.3.1 Corollary (Functional equation)**.** *The Jacobi theta function satisfies*[1]

$$\theta\left(-\frac{1}{z}\right) = \left(\frac{z}{i}\right)^{1/2}\theta(z).$$

[1] Here, $(z/i)^{1/2} = \exp((1/2)\log(z/i))$, taking the principal branch of the log.

Proof. Consider the function $\phi(t) := \exp(-\pi t^2)$, and observe that it is its own Fourier transform. Now for $x > 0$, write

$$\gamma_x(t) = \phi(\sqrt{x}t) = \exp(-\pi x t^2),$$

so that $\gamma_x(n) = g_n(x)$ in the notation above. Now

$$\widehat{\gamma_x}(y) = \frac{1}{\sqrt{x}}\widehat{\phi}\left(\frac{y}{\sqrt{x}}\right) = \frac{1}{\sqrt{x}}\phi\left(\frac{y}{\sqrt{x}}\right) = \frac{1}{\sqrt{x}}\exp\left(-\pi\frac{y^2}{x}\right) = \frac{1}{\sqrt{x}}\gamma_x\left(\frac{y}{x}\right).$$

Now we apply Poisson Summation to conclude. □

1.4.3.2 Corollary (Functional equation)**.** *One has*

$$Z(s) = Z(1-s).$$

Proof. Using the functional equation for θ, we obtain

$$f(x) = x^{-1/2} f\left(\frac{1}{x}\right) + \frac{1}{2}x^{-1/2} - \frac{1}{2},$$

and in the same manner as above, one may show that the Mellin transform of the function on the right is $M\{f\}(1/2-s)$. □

Exercise. Check that the Mellin transform behaves as described.

Let's use the same strategy to give a functional equation for the L-function of a nontrivial, primitive Dirichlet character χ of modulus k.

1.4.4 Definition. The *exponent* of χ is the quantity $\epsilon \in \{0,1\}$ with the property that

$$\chi(-1) = (-1)^{\epsilon}\chi(1).$$

Then the corresponding *theta series* is

When $m = \epsilon = 0$, we declare $m^{\epsilon} = 0$.

$$\theta(\chi, z) = \sum_{m\in\mathbf{Z}} \chi(m) m^{\epsilon} \exp\left(i\pi\frac{m^2}{k}z\right).$$

Let us run the same program as above. We study the Mellin transform of the function

$$f(\chi, x) = \frac{1}{2}\theta(\chi, ix),$$

and with the same argument as above, we obtain the following.

1.4.5 Proposition. *One has*

$$M\{f(\chi,\bullet)\}\left(\frac{s+\epsilon}{2}\right) = 2\left(\frac{k}{\pi}\right)^{(s+\epsilon)/2} L(\chi,s)\Gamma\left(\frac{s+\epsilon}{2}\right)$$

1.4.6 Definition. We define the *completed L-function* of χ:

$$\Lambda(\chi,s) \coloneqq \left(\frac{k}{\pi}\right)^{s/2} L(\chi,s)\Gamma\left(\frac{s+\epsilon}{2}\right)$$

We want to extract a functional equation for $\Lambda(\chi,s)$ from a functional equation for the theta series. This latter functional equation involves an auxiliary term called the *Gauß sum*.

1.4.7 Definition. The *Gauß sum* for χ is given by

$$\tau(\chi) = \sum_{m=0}^{k-1} \chi(m)\exp\left(i2\pi\frac{m}{k}\right).$$

One has

$$\begin{aligned} |\tau(\chi)|^2 &= \sum_{m=0}^{k-1}\sum_{n=0}^{k-1} \chi(n)\exp(i2\pi mn/k)\exp(-i2\pi m/k) \\ &= \sum_{n=0}^{k-1}\chi(n)\sum_{m=0}^{k-1}\exp(i2\pi m(n-1)/k) \\ &= \chi(1)m = m. \end{aligned}$$

with this in mind, here is our functional equation for the theta function.

1.4.8 Proposition (Functional equation)**.** *One has*

$$\theta\left(\chi,-\frac{1}{z}\right) = \frac{\tau(\chi)}{i^\epsilon\sqrt{m}}\left(\frac{z}{i}\right)^{\epsilon+1/2}\theta\left(\overline{\chi},z\right).$$

1.4.8.1 Corollary (Functional equation)**.** *One has*

$$\Lambda(\chi,s) = \frac{\tau(\chi)}{i^\epsilon\sqrt{m}}\Lambda(\overline{\chi},1-s).$$

We have a higher-dimensional variant of all this as well.

1.4.9 Notation. For any finite $\mathbf{Z}/2$-set S, and denote by σ the nontrivial involution. We will abuse notation and write σ also for complex conjugation. We form the $\mathbf{C}$-vector space

$$\mathbf{C}^S \coloneqq \mathrm{Map}(S,\mathbf{C})$$

with basis S, and we regard $\mathbf{C}^S$ as an algebra under pointwise addition and multiplication.

For $z \in \mathbf{C}^S$, we write

$$\operatorname{tr}(z) := \sum_{s \in S} z(s) \text{ and } N(z) := \prod_{s \in S} z(s).$$

There is an action of $\mathbf{Z}/2$ given by conjugation: $z \longmapsto \sigma \circ z \circ \sigma$, and we may identify the $\mathbf{R}$-algebra $\operatorname{Map}(S, \mathbf{R})$ with the fixed point subalgebra for this action:

$$\operatorname{Map}(S, \mathbf{R}) \cong \mathbf{R}^S := \operatorname{Map}(S, \mathbf{C})^{\mathbf{Z}/2} \subseteq \operatorname{Map}(S, \mathbf{C}).$$

This entitles us to speak of the *upper half-space*

$$\mathbf{H}^S := \left\{ z \in \mathbf{C}^S \,\middle|\, z = z \circ \sigma \text{ and } \frac{1}{2i}(z - \sigma \circ z \circ \sigma) > 0 \right\}.$$

1.4.10 Example. *A standard example of a suitable* $\mathbf{Z}/2$*-set* S *is the set* $\operatorname{Hom}(K, \mathbf{C})$ *for a number field* K*, with the obvious Galois action. Then one has isomorphisms*

$$\mathbf{C}^{\operatorname{Hom}(K,\mathbf{C})} \cong K \otimes_{\mathbf{Q}} \mathbf{C} \quad \textit{and} \quad \mathbf{R}^{\operatorname{Hom}(K,\mathbf{C})} \cong K \otimes_{\mathbf{Q}} \mathbf{R}.$$

1.4.11 Notation. There is also a hermitian form

$$(z, w) := \operatorname{tr}(z \cdot (\sigma \circ w)) = \sum_{s \in S} z(s) \cdot \sigma(w(s)),$$

which is invariant under the action of $\mathbf{Z}/2$. This restricts to an inner product $(\bullet, \bullet)$ on $\mathbf{R}^S$. Let μ denote the Haar meeasure relative to that metric.

1.4.12 Definition. For every $\mathbf{Z}$-structure $W \subset \mathbf{R}^S$, we define the *theta series*

$$\theta_W(z) = \sum_{w \in W} \exp\{i\pi(wz, w)\}.$$

This converges absolutely and uniformly on all compacta in $\mathbf{H}^S$.

More generally, suppose $a, b \in \mathbf{R}^S$, and suppose $p \colon S \longrightarrow \mathbf{N}_0$ such that $p \cdot (p \circ \sigma) = 0$. Set

$$\theta_W^p(a, b, z) = \sum_{w \in W} N\left((a + w)^p\right) \exp\{i\pi\left(((a + w)z, a + w) + 2(b, w)\right)\}.$$

We now contemplate Schwartz functions $\phi\colon \mathbf{R}^S \longrightarrow \mathbf{C}$ and their Fourier transforms

$$\widehat{\phi}(y) = \int_{\mathbf{R}^S} f(x) \exp(-i2\pi(x, y))\, d\mu,$$

and we see much of the good behaviour we've become accustomed to.

1.4.13 Example. *The function* $\phi(x) = \exp(-\pi(x, x))$ *is its own Fourier transform.*

To prove a functional equation for θ_W, we will need a Poisson Summation formula.

1.4.14 Proposition (Poisson Summation)**.** *If* $W \subset \mathbf{R}^S$ *is a* $\mathbf{Z}$*-structure, then let*

$$W^\vee \coloneqq \{v \in \mathbf{R}^S \mid \forall w \in W,\ (w, v) \in \mathbf{Z}\}.$$

Then for any Schwartz function $\phi\colon \mathbf{R}^S \longrightarrow \mathbf{C}$*, one has*

$$\sum_{w\in W} \phi(w) = \mathrm{covol}(W)^{-1} \sum_{v\in W^\vee} \widehat{\phi}(v).$$

If W is the $\mathbf{Z}$-span of vectors $w_1, \ldots, w_n$, then $\mathrm{covol}(W) \coloneqq |\det(w_1, \ldots, w_n)|$.

1.4.14.1 Corollary (Functional equation)**.** *One has*

$$\theta_W\left(-\frac{1}{z}\right) = \frac{\sqrt{(N(z/i))}}{\mathrm{covol}(W)}\theta_{W^\vee}(z).$$

More generally, one has

$$\theta_W^p\left(a, b, -\frac{1}{z}\right) = \left(i^{\mathrm{tr}(p)} \exp(i2\pi(a, b))\, \mathrm{covol}(W)\right)^{-1} N\left((z/i)^{p+1/2}\right) \theta_{W^\vee}^p(-b, a, z)\,.$$

1.4.15 Notation. We define

$$\left(\mathbf{R}^S\right)_{>0} \coloneqq \left\{x \in \mathbf{R}^S \mid x = x \circ \sigma \text{ and } x > 0\right\}.$$

One has a topological isomorphism

$$\phi\colon \left(\mathbf{R}^S\right)_{>0} \xrightarrow{\sim} \prod_{p\in S/C_2} \mathbf{R}_{>0}$$

defined by

$$x \longmapsto \left(\prod_{s\in p} x(s)\right)_{p\in S/C_2}.$$

(We say that p is *real* if it lies in the image of S^{C_2}; otherwise, we call it *complex*.) One now pulls back the Haar measure to obtain

$$d\log x \coloneqq \phi^\star\left(\prod_{s\in S} d\log x(s)\right).$$

1.4.16 Definition. The *Mellin transform* corresponding to the C_2-set S of a function $f\colon \left(\mathbf{R}^S\right)_{>0} \longrightarrow \mathbf{C}^S$ is the function

$$M_S\{f\}(s) \coloneqq \int_{x\in(\mathbf{R}^S)_{>0}} N(f(x)x^s)\, d\log x.$$

In particular, if $f(x) = exp(-x)$, then the *higher dimensional gamma function* is given by $\Gamma_S \coloneqq M\{f\}(s)$.

1.4.17 Proposition. *One obtains*

$$\Gamma_S(s) = \prod_{p\in S/C_2} \Gamma_p(s_p),$$

where

$$\Gamma_p(s_p) \coloneqq \begin{cases} \Gamma(s_p) & \text{if } p \text{ is real;} \\ 2^{1-\mathrm{tr}(s_p)}\Gamma(\mathrm{tr}(s_p)) & \text{if } p \text{ is complex.} \end{cases}$$

1.4.18 Definition. At last, we define the L-function of the C_2-set S by

$$L_S(s) \coloneqq N(\pi^{-s/2})\Gamma_S(s/2).$$

Using the proposition above, we find that

$$L_S(s) = \prod_{p\in S/C_2} L_p(s_p),$$

where

$$L_p(s_p) = \begin{cases} \pi^{-s_p/2}\Gamma(s_p/2) & \text{if } p \text{ is real;} \\ 2(2\pi)^{-\mathrm{tr}(s_p)/2}\Gamma(\mathrm{tr}(s_p)/2) & \text{if } p \text{ is complex.} \end{cases}$$

As a matter of notation, assume that $n \coloneqq \#S$, and

$$S = r_1(C_2/C_2) \cup r_2(C_2/e),$$

so that $n = r_1 + 2r_2$. We may apply any function of $\mathbf{C}^S$ to a single complex number s by writing $f(s) \coloneqq f(\mathrm{const}_s)$. So, for instance,

$$\begin{aligned} \Gamma_S(s) &= 2^{(1-2s)r_2}\Gamma(s)^{r_1}\Gamma(2s)^{r_2} \\ L_S(s) &= \pi^{-ns/2}\Gamma_S(s/2) \end{aligned}$$

In particular, for the two C_2-orbits,

$$\begin{aligned} L_{\mathbf{R}}(s) := L_{C_2/C_2}(s) &= \pi^{-s/2}\Gamma(s/2) \\ L_{\mathbf{C}}(s) := L_{C_2/e}(s) &= 2(2\pi)^{-s}\Gamma(s), \end{aligned}$$

and one has

$$L_S(s) = L_{\mathbf{R}}(s)^{r_1} L_{\mathbf{C}}(s)^{r_2}.$$

Using what we have already seen about the gamma function, we obtain the following directly:

1.4.19 Theorem. *One has*

$$\begin{aligned} L_{\mathbf{R}}(1) &= 1; \\ L_{\mathbf{C}}(1) &= \pi^{-1}; \\ L_{\mathbf{R}}(2+s) &= \frac{s}{2\pi} L_{\mathbf{R}}(s); \\ L_{\mathbf{C}}(1+s) &= \frac{s}{2\pi} L_{\mathbf{C}}(s); \\ L_{\mathbf{R}}(1-s)L_{\mathbf{R}}(1+s) &= \sec\left(\frac{\pi}{2}s\right); \\ L_{\mathbf{C}}(1-s)L_{\mathbf{C}}(s) &= 2\csc(\pi s); \\ L_S(s) &= \cos\left(\pi\frac{s}{2}\right)^{r_1+r_2} \sin\left(\pi\frac{s}{2}\right)^{r_2} L_{\mathbf{C}}(s)^n L_S(1-s). \end{aligned}$$

Now we apply this machinery to the C_2-set $S = \operatorname{Hom}(K, \mathbf{C})$ for a number field K of degree n over $\mathbf{Q}$. Then $\mathbf{R}^S$ is the Minkowski space, for which we have a selected isomorphism $\mathbf{R}^S \cong K \otimes_{\mathbf{Q}} \mathbf{R}$. Note that if $a \lhd O_K$, then a is a $\mathbf{Z}$-structure on $\mathbf{R}^S$ with covolume

$$\operatorname{covol}(a) = \sqrt{d_a},$$

where $d_a = N(a)^2|d_K|$ is the absolute value of the discriminant. Observe that if $x \in K^\times$, then

$$N((x)) = |N(x)|.$$

1.4.20 Definition. The *Dedekind zeta function* of K is given by

$$\zeta_K(s) := \sum_{0 \neq a \lhd O_K} N(a)^{-s},$$

where $N(a) := \#(O_K/a)$. This series converges on $\rangle 1, +\infty\langle$; moreover, one has

$$\zeta_K(s) = \prod_{0 \neq p \in \operatorname{Spec} O_K} (1 - N(p)^{-s})^{-1}.$$

For each ideal *class* $\Phi \in \mathrm{Cl}(K)$, one may define a *partial zeta function*

$$\zeta(\Phi, s) := \sum_{a \lhd O_K,\ a \in \Phi} N(a)^{-s},$$

so that

$$\zeta_K(s) = \sum_{\Phi \in \mathrm{Cl}(K)} \zeta(\Phi, s).$$

In effect, we are going to find functional equations for each partial zeta function, and then we are going to assemble these.

1.4.21 Notation. We write

$$Z(\Phi, s) := |d_K|^{s/2} L_X(s) \zeta(\Phi, s)$$

To express $Z(\Phi, s)$ as a suitable Mellin transform, we need to cut out the norm-one hypersurface

$$\mathbf{S}^S := \{x \in \left(\mathbf{R}^S\right)_{>0} \mid N(x) = 1\},$$

so that $\left(\mathbf{R}^S\right)_{>0} \cong \mathbf{S}^S \times \mathbf{R}_{>0}$. Of course $O_K^\times / \mu_K \subset \mathbf{S}^S$. We take $d^\times x$ to be the unique Haar measure on $\mathbf{S}^S$ such that

$$d \log x = d^\times x \times d \log t.$$

By the Dirichlet unit theorem,

$$\log(O_K^\times / \mu_K) \subset \{x \in \mathbf{R}^S \mid x = x \circ \sigma \text{ and } \operatorname{tr}(x) = 0\}$$

is a $\mathbf{Z}$-structure. Let F be the inverse image of any fundamental domain of $2 \log(O_K^\times / \mu_K)$.

1.4.22 Exercise. The domain F we constructed has volume

$$\mathrm{vol}(F) = 2^{r_1 + r_2 - 1} R_K, .$$

where

$$R_K := \frac{\mathrm{covol}(\log(O_K^\times / \mu_K))}{\sqrt{r_1 + r_2}}$$

is the quantity known as *Dirichlet regulator*.

1.4.23 Theorem. *Write*

$$f_F(a,t) \coloneqq \frac{1}{\#\mu_K}\int_F \theta_a(ix(t/d_a)^{1/n})\, d^\times x - \frac{\mathrm{vol}(F)}{\#\mu_K}.$$

Then $Z(\Phi, s) = M\{f\}(s/2)$.

Proof. Let R denote the quotient of the action of $O_K^\times$ upon the ideal a, and form the sum over representatives

$$g(x) = \sum_{r\in R} \exp(-\pi(rx/d_a^{1/n}, x)).$$

Then one has the following Exercise!

$$|d_K|^s \pi^{-ns}\Gamma_X(s)\zeta(\Phi, 2s) = \int_{(\mathbf{R}^S)_{>0}} g(x)N(x)^s\, d\log x.$$

Consequently, we obtain

$$Z(\Phi, 2s) = \int_{\mathbf{R}_{>0}} \left\{\int_{\mathbf{S}^S}\sum_{r\in R} \exp(-\pi(rx(t/d_a)^{1/n}, x))\, d^\times x\right\} t^s\, d\log t.$$

We'd like to connect this to the theta series, but this involves a sum over representatives of R rather than a sum over a itself. So:

$$\begin{aligned}
\int_{\mathbf{S}^S}\sum_{r\in R}\exp(-\pi(rx(t/d_a)^{1/n}, x))\, d^\times x &= \sum_{\eta\in|O_K^\times|}\int_{\eta^2F}\sum_{r\in R}\exp(-\pi(rx(t/d_a)^{1/n}, x))\, d^\times x \\
&= \frac{1}{\#\mu_K}\sum_{\eta\in O_K^\times}\int_{\eta^2F}\sum_{r\in R}\exp(-\pi(rx(t/d_a)^{1/n}, x))\, d^\times x \\
&= \frac{1}{\#\mu_K}\int_{\eta^2F}\sum_{\eta\in O_K^\times}\sum_{r\in R}\exp(-\pi(rx(t/d_a)^{1/n}, x))\, d^\times x \\
&= \frac{1}{\#\mu_K}\int_{\eta^2F}\left\{\theta_a(ix(t/d_a)^{1/n}) - 1\right\}\, d^\times x,
\end{aligned}$$

as desired. □

1.4.24 Theorem. *The function* $Z(\Phi, s)$ *admits a meromorphic continuation to* $\mathbf{C}$ *with simple poles at* 0 *and* 1 *with*

$$\mathrm{Res}_0\, Z(\Phi, s) = -\frac{2^{r_1+r_2}}{\#\mu_K}R_K \text{ and } \mathrm{Res}_1\, Z(\Phi, s) = \frac{2^{r_1+r_2}}{\#\mu_K}R_K.$$

Furthermore, it satisfies the functional equation

$$Z(\Phi, s) = Z(\Phi\otimes\omega, 1-s),$$

where ω is the codifferent ideal

$$\omega = \{x \in K \mid \operatorname{tr}_{K|\mathbf{Q}}(xO_K) \subseteq \mathbf{Z}\}.$$

Proof. In order to employ the functional equation for our θ-function, the first task is to sort out what the dual lattice to a is. Write $b = a^{-1} \otimes \omega$, so that $b = \{x \in \mathbf{R}^S \mid xa \in \omega\}$. One notes that

Exercise. Show that $d_b = 1/d_a$.

$$\begin{aligned} \overline{a^\vee} &= \{x \in \mathbf{R}^S \mid \operatorname{tr}(xa) \subseteq \mathbf{Z}\} \\ &= \{x \in \mathbf{R}^S \mid \forall r \in a,\ \operatorname{tr}_{K|\mathbf{Q}}(xrO_K) \subseteq \mathbf{Z}\} \\ &= \{x \in \mathbf{R}^S \mid xa \in \omega\} \\ &= b. \end{aligned}$$

Note also that $\theta_{\overline{a^\vee}} = \theta_{a^\vee}$, whence

$$\begin{aligned} f_F(a, \frac{1}{t}) &= \frac{1}{\#\mu_K} \int_F \theta_a(ix(td_a)^{-1/n})\, d^{times} x \\ &= \frac{1}{\#\mu_K} \frac{(td_a)^{1/2}}{\operatorname{covol}(a)} \int_{F^{-1}} \theta_b(ix(td_a)^{1/n})\, d^\times x \\ &= \frac{t^{1/2}}{\#\mu_K} \int_{F^{-1}} \theta_b(ix(t/d_b)^{1/n})\, d^\times x \\ &= t^{1/2} f_{F^{-1}}(b, t). \end{aligned}$$

Now the usual Mellin transform argument completes the proof. □

1.4.25 Definition. The *completed zeta function* of K is given by

$$Z_K(s) := |d_K|^{s/2} L_X(s) \zeta_K(s)$$

1.4.26 Theorem. *The function $Z_K(s)$ admits a meromorphic continuation to* $\mathbf{C}$ *with simple poles at* 0 *and* 1 *with*

$$\operatorname{Res}_0 Z(\Phi, s) = -\frac{2^{r_1+r_2} \# \operatorname{Cl}(K)}{\#\mu_K} R_K \text{ and } \operatorname{Res}_1 Z(\Phi, s) = \frac{2^{r_1+r_2} \# \operatorname{Cl}(K)}{\#\mu_K} R_K,$$

Furthermore, it satisfies the functional equation

$$Z_K(s) = Z_K(1 - s).$$

1.5 *Pontryagin duality and Fourier analysis*

BEFORE WE GO into details about Tate's thesis, it's good to have a crash course on locally compact abelian (LCA) groups and Pontryagin duality.

1.5.1 Example. *Here are some LCA groups.*

- *finite abelian groups – i.e., finite direct sums of cyclic groups* $\mathbf{Z}/p^{\nu}$ *– with the discrete topology*
- $\mathbf{Z}$ *with the discrete topology*
- $\mathbf{R}$ *with the usual topology*
- *any finite dimensional* $\mathbf{R}$*-vector space – these are called vector groups*
- $\mathbf{Q}$ *with the discrete topology (but not the subspace topology!)*
- *A closed subgroup of an LCA group is LCA.*
- *A quotient of an LCA group is LCA.*
- *The circle* $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ *is LCA.*
- *The adic circle* $\mathbf{T} = \mathbf{Q}/\mathbf{Z}$ *is LCA.*
- *Finite products and coproducts of LCA groups exist and coincide; we write* $\oplus$ *for the common operation. In particular,* **LCA** *is an additive category.*
- *If* $\{A_\alpha\}_{\alpha\in\Lambda}$ *is any collection of LCA groups, then the product*
$$\prod_{\alpha\in\Lambda} A_\alpha$$
is LCA.
- *The limit of any diagram of continuous homomorphisms of LCA groups is LCA.*
- *If* Φ *is the poset of natural numbers ordered by divisibility, then we obtain*
$$\widehat{\mathbf{Z}} \coloneqq \lim_{m\in\Phi^{op}} \mathbf{Z}/m;$$
we can look p*-typically as well:*
$$\mathbf{Z}_p \coloneqq \lim_{n\in\mathbf{N}_0^{op}} \mathbf{Z}/p^n.$$
These are rings.

Exercise. Characterise all compact hausdorff rings.

- *One has a topological isomorphism*

$$\widehat{\mathbf{Z}} \cong \prod_{p\in\Pi} \mathbf{Z}_p.$$

- *We can also form*

$$\widehat{\mathbf{Z}}^\times \coloneqq \lim_{m\in\Phi^{op}} (\mathbf{Z}/m)^\times;$$

  we can look p-typically as well:

$$\mathbf{Z}_p^\times \coloneqq \lim_{n\in\mathbf{N}_0^{op}} (\mathbf{Z}/p^n)^\times.$$

- *We have the solenoid*

$$\mathbf{S} \coloneqq \lim_{m\in\Phi^{op}} \mathbf{R}/m\mathbf{Z}$$

  and the p-solenoid

$$\mathbf{S}_p \coloneqq \lim_{n\in\mathbf{N}_0^{op}} \mathbf{R}/p^n\mathbf{Z}.$$

- *One has a topological isomorphism*

$$\mathbf{S}_p \cong (\mathbf{R}\times\mathbf{Z}_p)/\mathbf{Z}.$$

- *The filtered colimit of any diagram of open continuous homomorphisms of LCA groups is LCA.*

- *We have the finite adèles*

$$\mathbf{A}_{fin} \coloneqq \mathrm{colim}_{n\in\Phi}\, n^{-1}\widehat{\mathbf{Z}}.$$

  and the p-adics

$$\mathbf{Q}_p \coloneqq \mathrm{colim}_{n\in\mathbf{N}_0}\, p^{-n}\mathbf{Z}_p.$$

- *The adèles are given by*

$$\mathbf{A} \coloneqq \mathbf{R}\oplus\mathbf{A}_{fin}.$$

- *One has topological isomorphisms*

$$\mathbf{Q}/\mathbf{Z} \cong \mathrm{colim}_{m\in\Phi}\, \mathbf{Z}/m$$

  and

$$\mathbf{Q}_p/\mathbf{Z}_p \cong \mathrm{colim}_{n\in\mathbf{N}_0}\, \mathbf{Z}/p^n,$$

  both of which are discrete.

- *One has topological isomorphisms*

$$\mathbf{S} \cong \mathbf{A}/\mathbf{Q}.$$

$$\mathbf{S}_p \cong (\mathbf{R} \oplus \mathbf{Q}_p)/\mathbf{Z}[1/p].$$

- *If $\{A_\alpha\}_{\alpha\in\Lambda}$ is any collection of LCA groups, then the coproduct*

$$\coprod_{\alpha\in\Lambda} A_\alpha = \mathrm{colim}_{S\in\mathscr{F}(\Lambda)} \bigoplus_{\alpha\in S} A_\alpha,$$

where $\mathscr{F}(\Lambda)$ is the filtered poset of finite subsets of Λ, is LCA. ***Warning: the natural map $\coprod_{\alpha\in\Lambda} A_\alpha \longrightarrow \prod_{\alpha\in\Lambda} A_\alpha$ is continuous, and it is a set-theoretic inclusion, but it is not the inclusion of a subspace!***

- *Assume $\{A_\alpha\}_{\alpha\in\Lambda}$ is any collection of LCA groups, $J_\infty \subset \Lambda$ some finite subset, and $K_\beta \subseteq A_\beta$ a compact subgroup for each $\beta \in \Lambda - J_\infty$. Then the restricted product*

$$^{\{K_\beta\}}\prod_{\alpha\in\Lambda} A_\alpha := \mathrm{colim}_{S\in\mathscr{F}(\Lambda,J_\infty)} \left(\prod_{\alpha\in S} A_\alpha \times \prod_{\beta\in\Lambda-S} K_\beta\right),$$

where $\mathscr{F}(\Lambda, J_\infty)$ is the filtered poset of finite subsets of Λ that contain J_∞, is LCA. This is also the sum

$$^{\{K_\beta\}}\prod_{\alpha\in\Lambda} A_\alpha \cong \bigoplus_{\alpha\in J_\infty} A_\alpha \oplus \left(\left(\prod_{\beta\in\Lambda-J_\infty} A_\beta\right) \times_{\left(\prod_{\beta\in\Lambda-J_\infty}(A_\beta/K_\beta)\right)} \left(\coprod_{\beta\in\Lambda-J_\infty} (A_\beta/K_\beta)\right)\right).$$

Second warning: the natural map $^{\{K_\beta\}}\prod_{\alpha\in\Lambda} A_\alpha \longrightarrow \prod_{\alpha\in\Lambda} A_\alpha$ is continuous, and it is a set-theoretic inclusion, but it is not the inclusion of a subspace!

- *One has topological isomorphisms*

$$\mathbf{A}_{fin} \cong {}^{\{\mathbf{Z}_p\}}\prod_{p\in\Pi} \mathbf{Q}_p$$

and

$$\mathbf{A} \cong {}^{\{\mathbf{O}_{\mathbf{Q}_v}\}_{v\ finite}}\prod_{v} \mathbf{Q}_v,$$

where the restricted product is over the set of places, J_∞ is the set of infinite places, and $\mathbf{O}_{\mathbf{Q}_v} \subset \mathbf{Q}_v$ is the ring of integers.

- *More generally, for any number field K, one defines*

$$\mathbf{A}_K \cong {}^{\{\mathbf{O}_{K_v}\}_{v\ finite}}\prod_v K_v,$$

where J_∞ is the set of infinite places.

- *The multiplicative from of this are the idèles*

$$\mathbf{I}_K \cong {}^{\{\mathbf{O}_{K_v}^\times\}_{v\ finite}}\prod_v K_v^\times.$$

Warning: the natural map $\mathbf{I}_K \longrightarrow \mathbf{A}_K$ is continuous, and it is a set-theoretic inclusion, but it is not the inclusion of a subspace! *This is because multiplicative inverse on the suitable subspace of the adèles may not be continuous; we can repair this, however, and we can see that the inclusion $\mathbf{I}_K \hookrightarrow \mathbf{A}_K \times \mathbf{A}_K$ given by*

$$x \longmapsto (x, x^{-1})$$

is the inclusion of a subspace.

- *One has a topological isomorphism*

$$\mathbf{I}_\mathbf{Q} \cong \mathbf{Q}^\times \oplus \mathbf{R}_{>0} \oplus \widehat{\mathbf{Z}}^\times.$$

- *If K is a number field, then the K-solenoid is*

$$\mathbf{S}_K \coloneqq \mathbf{A}_K/K.$$

1.5.2 Definition. The *Pontryagin dual* of an LCA group A is the LCA group

$$\widehat{A} \coloneqq \mathrm{Hom}(A, \mathbf{T}).$$

The assignment $A \longmapsto \widehat{A}$ is a functor $\mathbf{LCA}^{op} \longrightarrow \mathbf{LCA}$.

1.5.3 Theorem (Pontryagin)**.** *The assignment $A \longmapsto \widehat{A}$ is an equivalence $\mathbf{LCA}^{op} \xrightarrow{\sim} \mathbf{LCA}$, and it is its own inverse.*

A	$\widehat{A}$
$\mathbf{Z}/p$	$\mathbf{Z}/p$
finite	*finite*
$\mathbf{Z}$	$\mathbf{T}$

discrete	*compact*
$\mathbf{R}$	$\mathbf{R}$
vector group V	$V^\vee$
product $\prod_{\alpha\in\Lambda} A_\alpha$	*coproduct* $\coprod_{\alpha\in\Lambda} \widehat{A}_\alpha$
limit $\lim A_\alpha$	*colimit* $\operatorname{colim} \widehat{A}_\alpha$
closed subgroup $B \le A$	*quotient* $\widehat{A}/B^\perp$
$\mathbf{Q}/\mathbf{Z}$	$\widehat{\mathbf{Z}}$
$\mathbf{Q}_p/\mathbf{Z}_p$	$\mathbf{Z}_p$
torsion	*profinite*
${}^{\{K_\beta\}}\prod_{\alpha\in\Lambda} A_\alpha$	${}^{\{K_\beta^\perp\}}\prod_{\alpha\in\Lambda} \widehat{A}_\alpha$
$\mathbf{A}_{fin}$	$\mathbf{A}_{fin}$
$\mathbf{Q}_p$	$\mathbf{Q}_p$
number field K	$\mathbf{A}_K/K$

1.5.4 Example.

We now give an abstract proof of Pontryagin duality in a manner that does not use the classification of LCA groups.

1.5.5 Definition. Suppose A a topological abelian group. We will say that A is *admissible* if it can be exhibited as a topological subgroup of a product $\prod_{\alpha\in\Lambda} B_\alpha$ of LCA groups.

Suppose A a topological abelian group. Then we say that a topology τ on $|A|$ is *A-characteristic* if $(|A|, \tau)$ is an admissible topological group, and

$$\operatorname{Hom}(A, \mathbf{T}) = \operatorname{Hom}((|A|, \tau), \mathbf{T}).$$

1.5.6 Proposition. *On any admissible topological abelian group A, there is both a coarsest and finest A-characteristic topology.*

Proof. The existence of the coarsest A-characteristic topology is an exercise; let C be the corresponding admissible topological abelian group. Now let $\{A'_\alpha\}_{\alpha\in\Lambda}$ be the collection of all admissible topological abelian groups such that $|A'_\alpha| = |A|$ and the topology on A'_α is A-characteristic. Form the pullback

$$\begin{array}{ccc} F & \longrightarrow & \prod_{\alpha\in\Lambda} A'_\alpha \\ \downarrow & & \downarrow \\ C & \xrightarrow[\Delta]{} & C^\Lambda. \end{array}$$

Of course $|F| = |A|$, and it is now a simple matter to see that the topology on F is A-characteristic. □

1.5.7 Definition. A topological abelian group is said to be $\mathbf{T}$-*cogenerated* if the topology on A is the finest A-characteristic topology. We write **TAb** for the category of $\mathbf{T}$-cogenerated topological abelian groups and continuous homomorphisms.

1.5.8 Exercise. Every LCA group is $\mathbf{T}$-cogenerated.

Here's an auxiliary category.

1.5.9 Definition. Denote by $\mathbf{D}$ the following category. The objects are pairs (A, B, η) consisting of (discrete) abelian groups A and B and a pairing $\eta\colon A \otimes B \longrightarrow \mathbf{T}$ that is *nondegenerate*. A morphism $(\phi, \psi)\colon (A, B, \eta) \longrightarrow (A', B', \eta')$ is a homomorphism $\phi\colon A \longrightarrow A'$ and a homomorphism $\psi\colon B' \longrightarrow B$ such that

Nondegeneracy means that for every $a \in A$, there is $b \in B$ such that $\eta(a, b) \neq 1$, and for every $b \in B$, there is $a \in A$ such that $\eta(a, b) \neq 1$.

$$\eta(\phi(a), b') = \eta(a, \psi(b')).$$

It is not difficult to see that this is an additive category. Furthermore, it is symmetric monoidal relative to the tensor product

$$(A, B, \eta) \otimes (A', B', \eta') = (A \otimes A', \underline{\mathrm{Hom}}(A, B') \times_{\underline{\mathrm{Hom}}(A\otimes A', \mathbf{T})} \underline{\mathrm{Hom}}(A', B), \xi).$$

Now the object $D \coloneqq (\mathbf{T}^{\delta}, \mathbf{Z}, id)$ is of particular import.

The following is now completely formal.

1.5.10 Proposition. *The category* $\mathbf{D}$ *is self-dual. More precisely, it is symmetric monoidal with respect to the tensor product above, it has an internal Hom* $\underline{\mathrm{Hom}}$*, and moreover the natural morphism*

$$A \longrightarrow \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(A, D), D)$$

is an isomorphism.

1.5.11 Theorem. *The category* **TAb** *is equivalent to* $\mathbf{D}$ *above, and moreover the functor* $\mathrm{Hom}(-, \mathbf{T})$ *on* **TAb** *coincides with the functor* $\underline{\mathrm{Hom}}(-, D)$ *on* $\mathbf{D}$.

Proof. The functor **TAb** $\longrightarrow$ **D** is given by the assignment

$$A \longmapsto (|A|, \mathrm{Hom}(A, \mathbf{T}), ev).$$

Its inverse is given as follows: for any (A, A', η), give A the subspace topology of $\widehat{A'}$. Now define **D** $\longrightarrow$ **TAb** as the assignment

$$(A, A', \eta) \longmapsto (|A|, \tau),$$

where, τ is the finest A-characteristic topology on $|A|$. □

1.5.12 Notation. For any LCA group A, denote by $\mathscr{M}(A)$ the set of all regular, countably additive, complex Borel measures on A. Denote by $\mathscr{H}(A)$ the set of all *Haar measures* – i.e., nontrivial, biinvariant, regular, countably additive, Borel measures – on A. Recall that $\mathscr{H}(A)$ is an $\mathbf{R}_{>0}$-torsor.

1.5.13 Definition. Suppose A an LCA group and $\mu \in \mathscr{M}(A)$. Then denote by $\widehat{\mu}\colon \widehat{A} \longrightarrow \mathbf{C}$ the function

$$\widehat{\mu}(\chi) \coloneqq \int_A \overline{\chi(a)}\, d\mu(a).$$

This is a uniformly continuous, bounded function, the *Fourier–Stieltjes trasform* of μ. If $\lambda \in \mathscr{H}(A)$, and if μ is absolutely continuous with respect to λ, then by Radon–Nikodym, $d\mu = f\, d\lambda$ for some $f \in L^1(A, \lambda)$. In this case, we write $\widehat{f}$ for $\widehat{\mu}$. This is the *Fourier transform* of f.

In the other direction, if $\mu \in \mathscr{M}(\widehat{A})$. Then denote by $\check{\mu}\colon A \longrightarrow \mathbf{C}$ the function

$$\check{\mu}(a) \coloneqq \int_A \chi(a)\, d\mu(\chi).$$

This is a bounded function, the *inverse Fourier–Stieltjes trasform* of μ. If $\lambda \in \mathscr{H}(\widehat{A})$, and if μ is absolutely continuous with respect to λ, then by Radon–Nikodym, $d\mu = f\, d\lambda$ for some $f \in L^1(\widehat{A}, \lambda)$. In this case, we write $\check{f}$ for $\check{\mu}$. This is the *inverse Fourier transform* of f.

1.5.14 Theorem. *The assignment $\mu \longmapsto \widehat{\mu}$ is an isomorphism*

$$\mathscr{M}(A) \cong \mathscr{C}_u(\widehat{A}),$$

where $\mathscr{C}_u(\widehat{A})$ denotes the space of uniformly continuous, bounded functions on $\widehat{A}$; in fact, it is an algebra isomorphism for the convolutions.

The assignment $f \longmapsto \hat{f}$ *is an isomorphism*

$$L^1(A, \lambda) \cong \mathscr{W}_0(\widehat{A}),$$

where $\mathscr{W}_0(\widehat{A}) \subset \mathscr{C}_0(\widehat{A})$ *is a dense subalgebra called the Wiener algebra.*

1.5.15 Lemma. *If* $\lambda \in \mathscr{H}(A)$*, then there is a* $\mu \in \mathscr{H}(\widehat{A})$ *such that for any* $f \in L^1(A, \lambda) \cap L^2(A, \lambda)$*, one has*

$$\hat{f} \in \mathscr{C}_0(\widehat{A}) \cap L^2(\widehat{A}, \mu), \text{ and } \|\hat{f}\|_2 \leq \|f\|_2.$$

Dually, for any $\phi \in L^1(\widehat{A}, \mu) \cap L^2(\widehat{A}, \mu)$*, one has*

$$\check{\phi} \in \mathscr{C}_0(A) \cap L^2(A, \lambda), \text{ and } \|\check{\phi}\|_2 \leq \|\phi\|_2.$$

1.5.16 Definition. If λ and μ are as in the previous Lemma, then since $L^1(A, \lambda) \cap L^2(A, \lambda) \subset L^2(A, \lambda)$ is dense (and similarly for $L^2(A, \lambda)$), the Fourier transform $f \longmapsto \hat{f}$ and the inverse Fourier transform $\phi \longmapsto \check{\phi}$ extend to continuoun linear maps

$$L^2(A, \lambda) \longrightarrow L^2(\widehat{A}, \mu) \text{ and } L^2(\widehat{A}, \mu) \longrightarrow L^2(A, \lambda),$$

respectively.

1.5.17 Theorem (Fourier Inversion/Plancherel)**.** *For any* $\lambda \in \mathscr{H}(A)$*, then there is a unique* $\mu \in \mathscr{H}(\widehat{A})$ *such that:*

(1) *for any* $f \in L^1(A, \lambda) \cap L^2(A, \lambda)$*, one has*

$$\hat{f} \in \mathscr{C}_0(\widehat{A}) \cap L^2(\widehat{A}, \mu), \text{ and } \|\hat{f}\|_2 \leq \|f\|_2;$$

(2) *dually, for any* $\phi \in L^1(\widehat{A}, \mu) \cap L^2(\widehat{A}, \mu)$*, one has*

$$\check{\phi} \in \mathscr{C}_0(A) \cap L^2(A, \lambda), \text{ and } \|\check{\phi}\|_2 \leq \|\phi\|_2;$$

(3) *for any* $f \in L^2(A, \lambda)$*,*

$$\check{\hat{f}} = f;$$

(4) *for any* $\phi \in L^2(\widehat{A}, \mu)$*,*

$$\hat{\check{\phi}} = \phi;$$

(5) *(Plancherel) The assignments* $f \longmapsto \hat{f}$ *and* $\phi \longmapsto \check{\phi}$ *are isometries, so that* $L^2(A, \lambda) \cong L^2(\widehat{A}, \mu)$*.*

The pair (λ, μ) of the theorem are called *dual Haar measures*. We'll get in the habit of writing λ for a Haar measure on A and $\hat{\lambda}$ for its dual Haar measure on $\widehat{A}$.

Here's a standard identity

1.5.18 Theorem (Parseval)**.** *For any $f, g \in L^2(A, \lambda)$, one has*

$$\int_A f\overline{g}\, d\lambda = \int_{\widehat{A}} \hat{f}\overline{\hat{g}}\, d\hat{\lambda}.$$

Now in order to get off the ground with a general Poisson summation formula, we must isolate a class of functions on any LCA group that are *of rapid decay*. These are called the *Schwartz–Bruhat functions*. Again, normally one proceeds by means of a classification result. We do things differently here, following Scott Osborne's paper.

1.5.19 Definition. A function $f \in L^\infty(A, \lambda)$ is *of brisk decay* if there exists a compactum $K \subset A$ such that for every integer $n \geq 1$ there is a constant $C_n > 0$ such that for any integer $m \geq 1$, one has

$$\|f|_{(A \setminus K^m)}\|_\infty < \frac{C_n}{m^n}.$$

1.5.20. Easy observations include:

(1) Brisk decay is independent of the choice of λ.

(2) If K is the compactum for a function f of brisk decay, then f vanishes ae away from the subgroup generated by K.

(3) Functions of brisk decay are translation-invariant.

(4) Functions of brisk decay are closed under convolution.

1.5.21 Exercise. Prove that every function of brisk decay lies in $L^p(A, \lambda)$ for every $p \geq 1$. (Hint: bound the integral of $|f|$ over $K^m \setminus K^{m-1}$.)

1.5.22 Definition. We say that a function $f \in L^\infty(A, \lambda)$ is *of rapid decay* – or is a *Schwartz–Bruhat function* – if and only if f is of brisk decay on A and $\hat{f}$ is of brisk decay on $\widehat{A}$. We write $\mathcal{S}(A)$ for the collection of Schwartz–Bruhat functions.

1.5.23 Lemma. *The vector space $\mathcal{S}(A)$ is a Fréchet space that is dense in $L^2(A, \lambda)$.*

I believe $\mathcal{S}(A)$ is also nuclear, but I didn't put any thought into this.

1.5.24 Lemma. *The Fourier transform gives a topological isomorphism $\mathcal{S}(A) \cong \mathcal{S}(\widehat{A})$.*

1.5.25 Lemma. *Every Schwartz–Bruhat function lies in $\mathscr{C}_0(A)$.*

1.5.26 Example. *On a finite group, all functions are Schwartz-Bruhat functions.*

1.5.27 Example. *On $\mathbf{Z}^m$, the Schwartz-Bruhat functions are precisely the functions f such that for any $k \in \mathbf{Z}^m_{>0}$, the quantity*

$$\sup_{n \in \mathbf{Z}^m} |n^k f(n)| < +\infty.$$

1.5.28 Example. *On $\mathbf{T}^m$, the Schwartz-Bruhat functions are precisely the smooth functions.*

1.5.29 Example. *On $\mathbf{R}^m$, the Schwartz-Bruhat functions are precisely the usual Schwartz functions.*

1.5.30 Exercise. Describe the Schwartz–Bruhat functions on any group of the form $F \times \mathbf{Z}^m \times \mathbf{T}^n \times \mathbf{R}^p$, where F is finite abelian.

1.5.31 Example. *One has*

$$\mathcal{S}(A) = \operatorname{colim}_{(U,K)} \mathcal{S}(U/K),$$

where the colimit is over subgroups $K < U < A$ with U open and compactly generated, K compact, and U/K a Lie group (and in particular of the form $F \times \mathbf{Z}^m \times \mathbf{T}^n \times \mathbf{R}^p$).

1.5.32 Example. *If A is totally disconnected, then $\mathcal{S}(A)$ is precisely the collection of locally constant functions with compact support.*

1.5.33 Theorem (Poisson summation)**.** *Suppose*

$$0 \to A' \to A \to A'' \to 0$$

a short exact sequence of LCA groups.

(1) *For any $\lambda \in \mathscr{H}(A)$ and $\lambda' \in \mathscr{H}(A')$, there exists a unique $\lambda'' \in \mathscr{H}(A'')$ such that for any $f \in \mathcal{S}(A)$,*

$$\int_{x\in A} f(x)\,d\lambda(x) = \int_{z\in A''}\int_{y\in A'} f(yz)\,d\lambda'(y)\,d\lambda''(z).$$

(2) *For any $f \in \mathcal{S}(A)$, define a function $\pi_*(f)$ on A'' by integration along the fibers*

$$\pi_*(f)(z) := \int_{y\in A'} f(xy)\,d\lambda''(y),$$

where x is any lift of z to A. Then on $\widehat{A''}$, which is canonically identified with $(A')^\perp$, we have

$$\widehat{\pi_*(f)} = \widehat{f}|_{(A')^\perp}.$$

(3) *For any $x \in A$,*

$$\int_{y\in A'} f(xy)\,d\lambda'(y) = \int_{\chi\in\widehat{A''}} \widehat{f}(\chi)\chi(x)\,d\widehat{\lambda''}(\chi)$$

Proof. The proof of point (1) is straightforward.[2]

For $\chi \in (A')^\perp$ and for any $x \in A$ and $y \in A'$, one has $\chi(xy) = \chi(x)$. Hence

$$\begin{aligned}\widehat{\pi_*(f)}(\chi) &= \int_{A''} \pi_*(f)(z)\overline{\chi(x)}\,d\lambda''(z)\\ &= \int_{A''}\int_{A'} f(yz)\overline{\chi(yz)}\,d\lambda'(y)\,d\lambda''(xA')\\ &= \int_A f(x)\overline{\chi(x)}\,d\lambda(x),\end{aligned}$$

where the last identification follow from point (1). This proves point (2).

Finally, Fourier Inversion gives, for any $x \in A$ with image $z = \pi(x)$,

$$\begin{aligned}\int_{y\in A'} f(xy)\,d\lambda'(y) &= \pi_*(f)(z) = \widecheck{\widehat{\pi_*(f)}}(z)\\ &= \widecheck{\widehat{f}|_{(A')^\perp}}(z)\\ &= \int_{\chi\in(A')^\perp} \widehat{f}(\chi)\chi(x)\,d\widehat{\lambda}(\chi),\end{aligned}$$

whence we obtain (3). □

[2] And consequently it is an Exercise.

1.5.33.1 Corollary. *If $\Lambda \subset A$ is a discrete cocompact subgroup, then for any $f \in \mathcal{S}(A)$, one has*

$$\sum_{x\in\Lambda} f(x) = \frac{1}{\mathrm{covol}(A/\Lambda)} \sum_{\chi\in\Lambda^\perp} \widehat{f}(\chi).$$

1.6 *Local zeta functions*

We now assume that k is a local field of characteristic 0.

1.6.1. We define an absolute value $|\cdot|$ on k by choosing a Haar measure λ, and a measurable subset[3] U of finite measure, and declaring

$$|x| := \frac{\mu(xU)}{\mu(U)}.$$

[3] A compact neighborhood of 0 will do.

The resulting absolute value does not depend on λ or U.

We have three options:

- If $k \cong \mathbf{R}$, then $|\cdots|$ is the ordinary abolute value.
- If $k \cong \mathbf{C}$, then $|\cdots|$ is the square of the ordinary absolute value.
- If k is nonarchimedean, let $o \subset k$ be the ring of integers, $p \lhd o$ the maximal ideal, π a uniformizing parameter, and F the finite residue field of order q. Then $|\pi| = 1/q$.

1.6.2. We have seen that k^+ is Pontryagin self-dual; the isomorphism $k^+ \cong \widehat{k^+}$ is not unique, but for any character $\chi \in \widehat{k^+}$, we may obtain such an isomorphism by $x \longmapsto \chi(x)$. For each such χ, there exists a unique Haar measure μ^+ on k^+ such that $\widehat{\mu^+} = \mu^+$ under this identification, and moreover, μ^+ does not depend on χ. We therefore dub this the *canonical additive* Haar measure.

1.6.3. It is convenient[4] to fix a character $X_k \in \widehat{k^+}$ – and hence an isomorphism $k^+ \cong \widehat{k^+}$ – once and for all. We have cases

[4] though perhaps unnecessary? I'm still not clear on how much really depends on these choices, which do seem in some sense uniform.

(1) If $k \cong \mathbf{R}$, then select

$$X_{\mathbf{R}} : \mathbf{R} \underset{(-1)}{\longrightarrow} \mathbf{R} \to \mathbf{R}/\mathbf{Z} \cong S^1.$$

(2) If $k \cong \mathbf{Q}_p$, then select

$$X_{\mathbf{Q}_p} : \mathbf{Q}_p \to \mathbf{Q}_p/\mathbf{Z}_p \subset \mathbf{Q}/\mathbf{Z} \subset S^1.$$

(3) If $k \cong \mathbf{F}_p((t))$, then take the coefficient of t^{-1} as a map $\mathbf{F}_p((t)) \longmapsto \mathbf{F}_p$ and select

$$X_{\mathbf{F}_p((t))} : \mathbf{F}_p((t)) \to \mathbf{F}_p \subset S^1.$$

(4) If $k \subset k_0$ is an extension of one of the above "prime" cases, then select

$$X_k := X_{k_0} \circ \mathrm{tr}_{k/k_0}.$$

1.6.4. Now we look to $k^\times$. We have a exact sequence

$$1 \to U_k \to k^\times \to V_k \to 1,$$

where

$$V_k := \{v \in \mathbf{R}_{>0} \mid \exists x \in k^\times \; v = |x|\}$$

is the *valuation group* of $k^\times$, which is $\mathbf{R}_{>0}$ itself if k is archimedean and $q^{\mathbf{Z}}$ otherwise, and

$$U_k := \{x \in k^\times \mid |x| = 1\}$$

is compact. This sequence is (noncanonically) split by a homomorphism $x \longmapsto \tilde{x}$, which in the nonarchimedean case is determined by a uniformising parameter.

Now we wish to specify a *canonical multiplicative Haar measure*. To this end, if $g \in \mathscr{C}_{00}(k^\times)$, then the function $g(x)/|x|$ is a function in $\mathscr{C}_{00}(k^+ - \{0\})$. So we define a functional Φ on $\mathscr{C}_{00}(k^\times)$ by

$$\Phi(g) := \int_{k^+ - \{0\}} \frac{g(x)}{|x|} \, d\mu^+.$$

1.6.5 Exercise. Show that the functional Φ is invariant under translation.

Consequently, Φ corresponds to a Haar measure which we might as well call $\log|\mu^+|$. Now we normalise $\log|\mu^+|$:

- If k is archimedean, then $\mu^\times = \log|\mu^+|$.
- If k is nonarchimedean, then

$$\mu^\times = \frac{q}{q-1} \log|\mu^+|.$$

1.6.6 Exercise. If k is the completion of a number field K at a finite place p, compute the volume of U_k under $\mu^\times$. Answer:

$$\mu^\times(U_k) = \frac{1}{\sqrt{d_k}},$$

where d_k is the discriminant of k, which is 1 for all p except those that divide the discriminant of K.

1.6.7 Definition. A *quasicharacter* on $k^\times$ is a homomorphism $k^\times \longrightarrow \mathbf{C}^\times$.

1.6.8. Any quasicharacter ψ on $k^\times$ factors as

$$\psi(x) = \chi(\tilde{x})|x|^s,$$

where χ is the character on U_k obtained as the restriction of ψ, and s is determined by ψ completely if k is archimedean, and modulo $2\pi i/\log q$ if k is nonarchimedean.

The assignment $\psi \longmapsto (\chi, s)$ identifies the set Q_k of quasicharacters on $k^\times$ with $\widehat{U_k} \times S_k$, where

$$S_k = \begin{cases} \mathbf{C} & \text{if } k \text{ is archimedean;} \\ \mathbf{C}/(2\pi i/\log q)\mathbf{Z} & \text{if } k \text{ is nonarchimedean.} \end{cases}$$

We endow Q_k with the structure of a complex manifold[5] by declaring that for any character[6] $\chi \in \widehat{U}_k$, the map $s \longmapsto \chi(\tilde{x})|x|^s$ should be a local isomorphism.

Also note that the quantity $\Re(\psi) := \Re(s)$ is uniquely determined by ψ.[7] We call $\Re(\psi)$ the *exponent* of s.

In effect, Q_k is going to be our analogue of the complex plane, and we're again going to work with "strips"

$$\rangle a, b\langle := \{\psi \in Q_k \mid \Re(\psi) \in]a, b[\}.$$

1.6.9 Definition. Suppose now $f \in \mathcal{S}(k^+)$. For any quasicharacter ψ with $\Re(\psi) > 0$, set

$$z(f, \psi) := \int_{k^\times} f(x)\psi(x)\, d\mu^\times(x).$$

1.6.10 Lemma. *The function $z(f, \psi)$ is well defined and holomorphic on the region $\rangle 0, +\infty\langle$.*

Now to get our local functional equation, we need the analogue of $s \longmapsto 1 - s$ on Q_k. This is the following.

[5] with infinitely many components

[6] Note that $\widehat{U}_k$ is discrete.

[7] Why?

1.6.11 Notation. Define an involution $\psi \longmapsto \widehat{\psi}$ on Q_k, where

$$\widehat{\psi}(x) := \frac{|x|}{\psi(x)}.$$

Note that $\mathbb{R}(\widehat{\psi}) = 1 - \mathbb{R}(\psi)$.

1.6.12 Exercise. Use Fubini to prove that for $\psi \in \rangle 0, 1\langle$ and $f, g \in \mathcal{S}(k^+)$, we have

$$z(f,\psi)z(\widehat{g},\widehat{\psi}) = z(g,\psi)z(\widehat{f},\widehat{\psi}).$$

For each character $\chi \in \widehat{U}_k$, if we can find **one** function $f_\chi \in \mathcal{S}(k^+)$ such that:

(1) $\psi \longmapsto z(\widehat{f}_\chi, \widehat{\psi})$ is not identically 0 on $(\{\chi\} \times S_k) \cap \rangle 0, 1\langle$, and

(2) the function

$$\rho(\psi) := \frac{z(f,\psi)}{z(\widehat{f},\widehat{\psi})}$$

admits a meromorphic continuation to all of $\{\chi\} \times S_k$,

then for **any** $g \in \mathcal{S}(k^+)$, the function $z(g,\psi)$ admits a meromorphic continuation to all of Q, and moreover

$$z(g,\psi) = \rho(\psi)z(\widehat{g},\widehat{\psi}).$$

We'll construct our functions f_χ case by case for the "prime fields." The appropriate mods involving the trace are left as an Exercise.

(1) Suppose $k = \mathbf{R}$. There are two options for χ:

(1.a) For $\chi = 1$, take $f_1(x) = \exp(-\pi x^2)$, so that when $\psi(x) = |x|^s$, we get

$$z(f_1, \psi) = \pi^{-s/2}\Gamma(s/2),$$

and

$$z(\widehat{f}_1, \widehat{\psi}) = \pi^{(s-1)/2}\Gamma(1 - s/2).$$

Hence

$$\rho(s) = 2^{1-s}\pi^{-s}\cos(\frac{\pi}{2}s)\Gamma(s),$$

which we've seen is meromorphic on $\mathbf{C}$.

(1.b) For $\chi = -1$, take $f_{-1}(x) = x\exp(-\pi x^2)$, so that when $\psi(x) = \mathrm{sgn}(x)|x|^s$, we end up with

$$\rho(s) = i2^{1-s}\pi^{-s}\sin(\frac{\pi}{2}s)\Gamma(s),$$

which again is meromorphic on $\mathbf{C}$.

(2) Suppose $k = \mathbf{Q}_p$. Then we have two options for χ:

(2.a) $\chi = 1$ (unramified case). Choose f_1 to be the indicator function of $\mathbf{Z}_p$. Then

$$\widehat{f_1}(x) = \int_{\mathbf{Z}_p} \exp(-2\pi i\{xy\})\, d\mu(y) = f_1.$$

Now let's perform our Mellin integrals for $\psi = |\cdot|^s$ (for $s \in \rangle 0, 1\langle$:

$$\begin{aligned} z(f_1,\psi) &= \int_{\mathbf{Q}_p^\times} f_1(x)|x|^s\, d\mu^\times(x) \\ &= \frac{p}{p-1}\int_{\mathbf{Z}_p-\{0\}} |x|^{s-1} d\mu(x) \\ &= \sum_{r=0}^{+\infty} p^{-rs} = \frac{1}{1-p^{-s}}, \end{aligned}$$

and, on the other side,

$$\begin{aligned} z(\widehat{f_1},\widehat{\psi}) &= \int_{\mathbf{Q}_p^\times} \widehat{f_1}(x)|x|^{1-s}\, d\mu^\times(x) \\ &= \frac{1}{1-p^{s-1}}, \end{aligned}$$

which is not identically zero on $(\{1\}\times S_k)\cap \rangle 0,1\langle$. So

$$\rho(s) = \frac{1-p^(s-1)}{1-p^s},$$

which meromorphically continues to the component $\{1\}\times S_k$.

(3) $\chi \neq 1$ (ramified case). Now χ factors through some $(\mathbf{Z}/p^n)^\times \longrightarrow S^1$, and assume that n is minimal with this property. Set

$$f_\chi(x) := \begin{cases} 0 & \text{if } |x| > p^n; \\ \exp(2\pi i\{x\}) & \text{if } |x| \le p^n. \end{cases}$$

Basic properties of Fourier transforms give

$$\widehat{f}_\chi(x) = \begin{cases} 0 & \text{if } |1-x| > p^n; \\ p^n & \text{if } |1-x| \le p^n. \end{cases}$$

Now we have $\psi(x) = \chi(\tilde{x})|x|^s$, and we compute[8]

$$z(f_\chi, \psi) = \frac{p^{ns+1-n}}{p-1} \sum_{r=1}^{p^n-1} \chi(r) \exp(2\pi i r/p^n),$$

and

$$z(f_\chi, \psi) = \frac{p}{p-1},$$

and now

$$\rho(\psi) = p^{n(s-1)} \sum_{r=1}^{p^n-1} \chi(r) \exp(2\pi i r/p^n).$$

No problem continuing this to $\{1\} \times S_k$.

(4) When $K = \mathbf{F}_p((t))$, we'll run essentially the same program, with the indicator function in the unramified case and, in the ramified case, a rescaled version thereof multiplied by the character.

[8] Here's our old friend the Gaußsum!

1.7 *Global L-functions*

Now we let K be a global field. Our identifications $K_v^+ \cong \widehat{K_v^+}$ give us a selected identification

$$\mathbf{A}_K \cong \widehat{\mathbf{A}}_K.$$

There's a unique norm on this ring induced by the finite ones. Similarly, the *Tamagawa measure* on $\mathbf{A}_K$ is the measure μ^+ given by forming the product of the μ_v^+'s, and the *Tamagawa measure* on $\mathbf{I}_K$ is the measure $\mu^\times$ given by forming the product of the $\mu_v^\times$'s.

On $K \subset \mathbf{A}_K$, we install the counting measure. We have the following consequence of Poisson summation:

1.7.1 Proposition. $\mu(\mathbf{A}_K/K) = 1$.

For the idèles, we have a canonical norm coming from all the local norms, and to embed $k^\times$ nicely, we first need to pass to the idèles of norm 1:

$$1 \to \mathbf{I}_K^1 \to \mathbf{I}_K \to V_K \to 1.$$

Now if K is a number field, $V_K = \mathbf{R}_{>0}$, and if K is a function field, then $V_K = q^{\mathbf{Z}}$. In the first case, choose the measure on V_K to be $\log\mu$, and in the second, choose the measure to be $(\log q)\cdot\#$. Now select the measure μ^1 on $\mathbf{I}_K^\times$ to be compatible with the other two in the usual way.

1.7.2 Proposition. *$K^\times \subset \mathbf{I}_K^1$ is discrete and cocompact.*

1.7.3 Definition. A homomorphism $\chi\colon \widehat{\mathbf{I}_K/K^\times} \longrightarrow \mathbf{C}^\times$ is called a *quasicharacter*. We'll call the space of all these $\mathbf{H}$.

$\mathbf{H}$ decomposes (topologically) as the product $(K^\times)^\perp \times \mathbf{C}$; any $\psi \in \mathbf{H}$ can be written uniquely as $x \longmapsto \chi(x)|x|^s$, where $\chi \in (K^\times)^\perp$ and $s \in \mathbf{C}$ is an action of $\mathbf{C}$ that gives $\mathbf{H}$ a complex manifold structure.

Again any ϕ gives rise to $\widehat{\psi}$, given by $x \longmapsto \frac{|x|}{\psi(x)}$.

1.7.4 Definition. Suppose $f \in \mathcal{S}(\mathbf{A}_K)$, and suppose $\chi \in \mathbf{H}$. Then we define

$$Z(f,\chi) := \int_{\mathbf{I}_K} f(x)\chi(x)\,d\mu^\times(x).$$

1.7.5 Theorem. *The integral $Z(f,\psi)$ converges for $\Re(\psi) > 1$. It extends to a meromorphic function on $\mathbf{H}$, and the only poles are at $(1,0)$ with residue $-\operatorname{covol}(K^\times)f(0)$ and at $(1,1)$ with residue $\operatorname{covol}(K^\times)\widehat{f}(0)$. Finally, we have the simple functional equation*

$$Z(f,\psi) = Z(\widehat{f},\widehat{\psi}).$$

High points of proof. We can decompose that integral

$$Z(f,\chi) := \int_{\mathbf{I}_K} f(x)\chi(x)\,d\mu^\times(x) = \int_0^{+\infty} Z_t(f,\chi)\,d\log t,$$

where

$$Z_t(f,\chi) = \int_{\mathbf{I}_K^1} f(ty)\psi(ty)\,d\mu^1(y).$$

Now let's analyse Z_t *not* by decomposing it into local pieces, but instead selecting a fundamental domain E for $k^\times$ and writing

$$Z_t(f,\psi) = \int_E \left(\sum_{\alpha\in K^\times} f(\alpha t y)\right)\psi(ty)\,d\mu^1(y).$$

On the other hand, Poisson summation yields

$$\int_E \left(\sum_{\alpha \in K} f(\alpha t y) \right) \psi(ty)\, d\mu^1(y) = \int_E \left(\sum_{\alpha \in K} \widehat{f}(\alpha y/t) \right) \widehat{\psi}(y/t)\, d\mu^1(y),$$

whence one obtains

$$Z_t(f, \psi) + f(0) \int_E \psi(ty)\, d\mu^1(y) = Z_{1/t}(\widehat{f}, \widehat{\psi}) + \widehat{f}(0) \int_E \widehat{\psi}(y/t)\, d\mu^1(y)$$

It turns out that there are two options for those integrals: either the quasicharacter is unramified or it isn't. If it is, then we get

$$\int_E \psi(ty)\, d\mu^1(y) = t^s \mu^1(E);$$

if not, then we get

$$\int_E \psi(ty)\, d\mu^1(y) = 1.$$

So the ramified case is going to be easier: you take $Z(f, \psi)$ and split it into two pieces, using what you've learned.

$$Z(f, \chi) = \int_1^{+\infty} Z_t(f, \chi)\, d\log t + \int_1^{+\infty} Z_u(\widehat{f}, \widehat{\chi})\, d\log u.$$

That's literally it in that case.

The unramified case is only mildly more annoying. □

2

The field with one element

2.1 Borger's picture

Jim Borger has proposed a fascinating picture of $\mathbf{F}_1$, which offers an amazing insight into this object. To describe it, it's convenient to start with an understanding of *plethystic algebra*.

2.1.1. Our *rings* and *algebras* will all be commutative with 1.

2.1.2 Definition. Let k denote a ring. An *affine k-algebra scheme* is an endofunctor $X\colon \mathbf{Alg}_k \longrightarrow \mathbf{Alg}_k$ such that the composite

$$\mathbf{Alg}_k \to \mathbf{Alg}_k \to \mathbf{Set}$$

is corepresentable. If X is in addition a comonad, then we say that X is a *k-plethory*.

2.1.3. An affine k-algebra scheme is $X = \operatorname{Spec} R$ for a k-algebra R that has been equipped with the following structure:

- a cozero $\epsilon^+\colon R \longrightarrow k$;
- a coaddition $\Delta^+\colon R \longrightarrow R \otimes_k R$;
- an antipode $\sigma\colon R \longrightarrow R$;
- a coone $\epsilon^\times\colon R \longrightarrow k$;
- a comultiplication $\Delta^\times\colon R \longrightarrow R \otimes_k R$;
- a ring map $k \longrightarrow \operatorname{Hom}_k(R, k)$;

subject to a whole pile of axioms.

The structure of a plethory on X then suppiles one more piece of data on R, a noncommutative operation

$$\circ\colon\ R \times_k R \longrightarrow R$$

that corepresents the comonad structure. This is called the *composition product* or *plethysm*.

2.1.4 Example. *The constant endofunctor at the zero ring is an affine k-algebra scheme; it is corepresented by* $\operatorname{Spec} k$.

2.1.5 Example. *The identity endofunctor* $I\colon\ \mathbf{Alg}_k \longrightarrow \mathbf{Alg}_k$ *is an affine k-algebra scheme; it is corepresented by* $\operatorname{Spec} k[t]$. *Since it is obviously a comonad, this is also a plethory.*

2.1.6 Example. *The assignment* $S \longmapsto \mathbf{W}(S)$ *of the ring of "big" Witt vectors is a* $\mathbf{Z}$*-plethory. It is corepresented by the algebra* Λ *of symmetric functions over* $\mathbf{Z}$. *This is defined as the subring of* $\mathbf{Z}[\![x_1, x_2, \dots]\!]^{\operatorname{Aut}\mathbf{N}}$ *consisting of those power series in which the degree of the monomials is bounded. The theorem of elementary symmetric functions shows that*

$$\Lambda = \mathbf{Z}[\lambda_1, \lambda_2, \dots],$$

where

$$\lambda_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}$$

are the usual elementary symmetric functions. (We usually let $\lambda_0 = 1$.*)*

We could also freely generate Λ *under the complete symmetric functions*

$$\sigma_n = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}$$

Of special relevance are the Adams symmetric functions

$$\psi_n \coloneqq x_1^n + x_2^n + \cdots;$$

we may freely generate Λ *by elements* $w_1, w_2, \dots \in \Lambda$ *determined by the relations*

$$\psi_n = \sum_{d|n} d w_d^{n/d},$$

since

$$\sum_{m\in\mathbf{N}_0}(-1)^m\lambda_m T^m = \exp\left(-\sum_{m\in\mathbf{N}}\frac{1}{m}\psi_m t^m\right) = \prod_{m\in\mathbf{N}}(1-w_m t^m).$$

This is how one obtains the usual description of elements of $\mathbf{W}(S)$ *in terms of Witt components* $(w_1, w_2, \dots)$.

The plethysm is composition of symmetric functions; this corresponds to the Artin–Hasse map $\mathbf{W}(S) \longrightarrow \mathbf{W}(\mathbf{W}(S))$.

We will describe all these structures very precisely with K-theory soon.

2.1.7 Example. *If G is a group, then the endofunctor* $F^G\colon S \longmapsto S^G$ *has a comonad structure, and it is corepresented by* $k[G]$, *which is the free polynomial algebra on the underlying set of G. This is not the group-algebra, but it is the symmetric algebra generated by the group algebra kG.*

2.1.8 Definition. If P is a k-plethory, then a *P-algebra* is a coalgebra for the comonad P.

2.1.9 Example. *Every k-algebra is an I-algebra for the identity plethory I represented by* $k[x]$ *in a unique way.*

2.1.10 Example. *If G is a group, then an* F^G*-algebra is a k-algebra with an action of G.*

2.1.11 Definition. Suppose K an idempotent complete symmetric monoidal category with finite colimits (such that the symmetric monoidal structure preserves finite colimits separately in each variable). We write $\mathbf{Alg}(K)$ for the category of idenmpotent (commutative) K-algebras. A *K-biring* is a comonad $X\colon \mathbf{Alg}(K) \longrightarrow \mathbf{Alg}(K)$ such that the composite

$$\mathbf{Alg}(K) \to \mathbf{Alg}(K) \to \mathbf{Cat}$$

is corepresentable. If X is a comonad, then we say that X is a *K-plethory*.

[...]

2.1.12 Definition. A quotient of the plethory $\mathbf{W}$ corepresented by Λ is the subplethory $\mathbf{W}_\Psi$ corepresented by the subring $\Psi \subset \Lambda$ generated by the elements ψ_n for $n \in \mathbf{N}$.

2.1.13. The plethory $\mathbf{W}_\Psi$ is freely generated by the elements ψ_n for $n \in \mathbf{N}$; that is, for any ring R, one has

$$\mathbf{W}_\Psi(R) = R^{\mathbf{N}}$$

as a ring. For any $n \in \mathbf{N}$, define

$$\psi_n : R^{\mathbf{N}} \longrightarrow R^{\mathbf{N}}$$

by

$$\psi_n(\mathbf{a}) = \psi_n(a_1, a_2, a_3, \dots) = (a_n, a_{2n}, a_{3n}, \dots).$$

The comonad structure $\mathbf{W}_\Psi \longrightarrow \mathbf{W}_\Psi \circ \mathbf{W}_\Psi$ is then given by

$$(\mathbf{a}) \longmapsto (\psi_1(\mathbf{a}), \psi_2(\mathbf{a}), \psi_3(\mathbf{a}) \dots).$$

A Ψ-algebra is thus a ring R with an action of $\mathbf{N}^\times$. Of course and Λ-algebra gives rise to a Ψ-algebra structure.

To some degree, the structure of a Λ-algebra can be recovered from the strecture of a Ψ-algebra.

2.1.14 Lemma (Newton formula)**.** *For any $k \geq 1$, one has*

$$\sum_{m=1}^{k} (-1)^{k-m} \lambda_{k-m} \psi_k = (-1)^{k+1} k \lambda_k.$$

2.1.14.1 Corollary. *If R is flat over $\mathbf{Z}$, then for any Ψ-algebra structure on R, there is at most one Λ-algebra structure that lifts it.*

Proof. The lemma above permits one to write each λ_k as a homogeneous polynomial of degree k with $\mathbf{Q}$ coefficients in $\psi_1, \dots, \psi_k$, where $|\psi_j| = j$. □

2.1.15 Theorem (Wilkerson)**.** *If R is flat over $\mathbf{Z}$, then any Ψ-algebra structure on R in which for every prime p,*

$$\psi_p(x) \cong x^p \mod pR$$

admits a unique Λ-algebra structure lifting it.

Proof. We're just after existence. Uniqueness we already have. On $R \otimes \mathbf{Q}$, define λ_k for $k \geq 1$ by means of generating functions:

$$-t\frac{d}{dt}\log\left(\sum_{m\in\mathbf{N}_0}\lambda_m(x)t^m\right) = \sum_{n\in\mathbf{N}}(-1)^n\psi_n(x)t^n.$$

This is[1] a Λ-structure on $R \otimes \mathbf{Q}$.

[1] Exercise.

We aim to show that for any prime p and any $x \in R \otimes \mathbf{Z}_{(p)}$, the map $\lambda_k \colon R \otimes \mathbf{Q} \longrightarrow R \otimes \mathbf{Q}$ carries x to $R \otimes \mathbf{Z}_{(p)}$. This is obvious for $k = 1$, since $\lambda_1(x) = x$. Now assume the claim for all λ_m for $m < k$.

Suppose $p \nmid k$. Then we can divide by k, so the Newton formula plus the induction hypothesis does the job.

If $k = p$, we observe that

$$\psi_p(x) - x^p = p\left((-1)^{p+1}\lambda_p(x) + P(\lambda_1(x), \dots, \lambda_{p-1}(x))\right)$$

for some polynomial P with $\mathbf{Z}$ coefficients. By assumption, the left hand side lies in $p(P \otimes \mathbf{Z}_{(p)})$, so

$$(-1)^{p+1}\lambda_p(x) + P(\lambda_1(x), \dots, \lambda_{p-1}(x)) \in R \otimes \mathbf{Z}_{(p)}.$$

Now the induction hypothesis does the job.

If $k = mp$ for some m, then we observe that

$$\lambda_k(x) = (-1)^{(p+1)(m+1)}\lambda_m(\lambda_p(x)) + Q(\lambda_1(x), \dots, \lambda_{k-1})$$

for some polynomial Q with $\mathbf{Z}$ coefficients. Again with the induction hypothesis. □

2.1.16. Wilkerson's theorem says that if R is flat over $\mathbf{Z}$, then a Λ-algebra structure on R is exactly the same as a compatible family of lifts of frobenius for each prime p. Incidentally, one can show that if R is a reduced Λ-algebra, then it is automatically flat over $\mathbf{Z}$.

2.1.17 Definition (Borger)**.** An $\mathbf{F}_1$-*algebra* is a Λ-algebra. We write $\mathbf{CAlg}_{\mathbf{F}_1}$ for the category of such.

2.1.18. We have an adjunction

$$-\otimes_{\mathbf{F}_1}\mathbf{Z} \colon \mathbf{CAlg}(\mathbf{F}_1) \rightleftarrows \mathbf{CAlg}(\mathbf{Z}) \colon U,$$

where $-\otimes_{\mathbf{F}_1}\mathbf{Z}$ is the functor that forgets the Λ-algebra structure, and U is $\mathbf{W}$. The functor $-\otimes_{\mathbf{F}_1}\mathbf{Z}$ preserves both limits and colimits, since it admits a right adjoint ($\mathbf{W}$) and a left adjoint.

We think of the Λ-algebra structure as descent data to $\mathbf{F}_1$. When a ring R admits a Λ structure, we say that R *is defined over* $\mathbf{F}_1$ or *descends to* $\mathbf{F}_1$.

2.1.19 Example. *The ring* $\mathbf{Z}$ *is defined over* $\mathbf{F}_1$ *in a unique fashion, in which each* $\psi_p = id$.

2.1.20 Example. *For any commutative monoid* X*, the monoid algebra* $\mathbf{Z}X$ *has is defined over* $\mathbf{F}_1$*: the descent data are given by* $\psi_p(x) = x^p$ *for any* $x \in X$.

In this way, one has

$$\mathbf{Z}[t]/(t^n - 1) \cong \mathbf{F}_{1^n} \otimes_{\mathbf{F}_1} \mathbf{Z}.$$

The $\mathbf{F}_1$*-algebra* $\mathbf{F}_{1^n}$ *is the field with* 1^n *elements, which we regard as a field extension of* $\mathbf{F}_1$.

Pushing this farther, we can write

$$\mathbf{Z}(\mathbf{Q}/ZZ) \cong \overline{\mathbf{F}_1} \otimes_{\mathbf{F}_1} \mathbf{Z}.$$

2.1.21 Example. *There is actually another* $\mathbf{F}_1$*-structure on* $\mathbf{Z}[x]$ *(different from* $\mathbf{Z}[\mathbf{N}_0]$*). This is the one that has the potential to be more interesting from our point of view.*

Consider the category $\mathbf{Rep}(\mathbf{SL}_2(\mathbf{C}))$ *of finite dimensional representations of* $\mathbf{Rep}(\mathbf{SL}_2(\mathbf{C}))$*; denote by* V *the standard* 2*-dimensional representation. We have the representation ring* $K_0(\mathbf{Rep}(\mathbf{SL}_2(\mathbf{C})))$*, which is a* Λ*-algebra in the canonical manner. We may compute it by identifying a representation with its character on the torus elements* $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$*, so that* $[V] = a + a^{-1}$*. We find that* $K_0(\mathbf{Rep}(\mathbf{SL}_2(\mathbf{C})))$ *is then the fixed points of the ring* $\mathbf{Z}[a, a^{-1}]$ *under the* C_2 *action* $a \longmapsto a^{-1}$*. Of course, if* $x = [V] = a + a^{-1}$*, then we obtain an isomorphism*

$$K_0(\mathbf{Rep}(\mathbf{SL}_2(\mathbf{C}))) \cong \mathbf{Z}[x],$$

but the Λ*-algebra structure turns out to be different.*

Indeed, to see this, consider the 2^n*-dimensional tensor power* $V^{\otimes n}$*, which corresponds to* $[V^{\otimes n}] = x^n = (a + a^{-1})^n \in \mathbf{Z}[a, a^{-1}]$*, and the*

$(n+1)$*-dimensional representation* $\mathrm{Sym}^n(V)$*, which corresponds to*

$$[\mathrm{Sym}^n(V)] = a^n + a^{n-2} + \cdots + a^{2-n} + a^n \in \mathbf{Z}[a + a^{-1}].$$

If we substitute $a = \exp(i\theta)$*, then we get* $x = 2\cos\theta$*, and we find that*

$$[\mathrm{Sym}^n(V)] = U_n(x/2),$$

where U_n *are the Chebyshev polynomials of the second kind. Equivalently, the Adams operations are given by*

$$\psi_p(x) = 2T_p(x/2),$$

where T_p *are the Chebyshev polynomials of the first kind.*

2.1.22. One might like to declare that the "correct" polynomial ring in one variable over $\mathbf{F}_1$ is the free $\mathbf{F}_1$-algebra generated by one element. The problem is that it's not so clear what "free" means, and to this end the problem is that it's not so clear what the "underlying set" of an $\mathbf{F}_1$-algebra ought to be.

To illustrate, consider a finite Galois extension $F \subset E$ with group G. Then an F-algebra is the "same" as an E-algebra R equipped with a semilinear action of G. The underlying set of the F-algebra is, of course, not the underlying set of R, but rather the underlying set of R^G. Under the analogy between Λ and $E^\sigma[G]$, it's not a priori obvious what should play the role of R^G. In the particular case in which R is flat, a reasonable answer might be fixed points for the action of $\mathbf{N}^\times$.

To describe the theory of more general objects over $\mathbf{F}_1$, we can extend the comonad $\mathbf{W}$ as follows.

2.1.23 Definition. Consider $\mathbf{Shv}(\mathbf{Z})$, the category of étale sheaves (of sets) on the category $\mathbf{Aff}/\mathbf{Z}$ of finitely generated (over $\mathbf{Z}$) affine schemes. Since $\mathbf{W}$ is a monad on $\mathbf{Aff}/\mathbf{Z}$, the left Kan extension of $\mathbf{W}$ is again a monad on $\mathbf{Shv}(\mathbf{Z})$, and we write $\mathbf{Shv}(\mathbf{F}_1)$ for the resulting category of algebras, and we call the objects thereof $\mathbf{F}_1$-sheaves.

Observe that $\mathbf{Shv}(\mathbf{F}_1)$ can equally well be considered the category of coalgebras over the comonad given by precomposition with $\mathbf{W}$. Furthermore, it is not at all difficult to check that precomposition with $\mathbf{W}$ preserves filtered colimits, whence $\mathbf{Shv}(\mathbf{F}_1)$ is a topos, which we call the *large étale topos of* $\mathbf{F}_1$. Furthermore, the adjunction

$$v^* : \mathbf{Shv}(\mathbf{F}_1) \rightleftarrows \mathbf{Shv}(\mathbf{Z}) : v_*$$

is a geometric morphism of topoi. In fact, it is an *essential geometric morphism* – the left adjoint v^* admits a further left adjoint $v_!$. However, this geometric morphism is not étale, since the left adjoint $v_!$ is not conservative; indeed, this would follow if and only if the assignment $R \longmapsto \mathbf{W}(R)$ is conservative.

We have to contend with the theory of modules over $\mathbf{F}_1$-algebras. Luckily, we can be surprisingly concrete.

2.1.24 Definition (Beck)**.** If C is a category that admits all finite limits, then for any object $A \in C$, one may define an *A-module* to be an abelian group object in the category $C_{/A}$, so that

$$\mathbf{Mod}(A) := \mathbf{Ab}(C_{/A}).$$

If M is an A-module in this sense, then a *derivation* from A to M is a morphism $A \longrightarrow M$ in $C_{/A}$; the set

$$\mathrm{Der}(A, M) := \mathrm{Mor}_{C_{/A}}(A, M)$$

admits an abelian group structure since M does.

2.1.25 Exercise. This recovers the usual notion of a module when C is the category of rings. Indeed, for any object $B \in \mathbf{Ab}(C_{/A})$, the kernel $\ker[B \longrightarrow A]$ inherits an A-module structure in the usual sense. On the other hand, if M is an A-module in the usual sense, endow $A \oplus M$ with the ring structure of square-zero extension of A. Show that these assignments define inverse equivalences of categories.

2.1.26 Proposition. *If C is presentable, then the forgetful functor*

$$U : \mathbf{Mod}(A) \longrightarrow C_{/A}$$

admits a left adjoint Ω. Consequently, one has a universal derivation $d : A \longrightarrow \Omega_A$ that induces an isomorphism

$$\mathrm{Mor}_{\mathbf{Mod}(A)}(\Omega_A, M) \cong \mathrm{Der}(A, M).$$

2.1.27 Definition. It is a simple matter to see that the comonad $\mathbf{W}$ extends to a comonad on the category $\mathbf{CAlg}^{nu}(\mathbf{Z})$ of non-unital rings. In particular, any abelian group can fed into $\mathbf{W}$ as a ring with zero multiplication.

If A is a ring and M is an A-module, then we write $\mathbf{W}(M)$ for the $\mathbf{W}(A)$-module given by the set $M^{\mathbf{N}}$ with componentwise addition and scalar multiplication given by

$$(am)_k \coloneqq \psi_k(a)m_k.$$

If A is an Λ-ring, then a *Λ-module over A* is an A-module M and an A-linear map $\lambda\colon M \longrightarrow \mathbf{W}(M)$ such that $\epsilon \circ \lambda = id$, and the square

$$\begin{array}{ccc} M & \longrightarrow & \mathbf{W}(M) \\ \downarrow & & \downarrow {\scriptstyle \mathbf{W}(\lambda)} \\ \mathbf{W}(M) & \underset{\Delta}{\longrightarrow} & \mathbf{WW}(M) \end{array}$$

2.1.28 Proposition (Hesselholt). *Suppose A an $\mathbf{F}_1$-algebra. Then the category $\mathbf{Mod}(A)$ is equivalent to the category of Λ-modules on A.*

Proof. Formal. The assignment $B \longmapsto \ker[B \to A]$ lifts to an equivalence from $\mathbf{Mod}(A)$ to Λ-modules on A. □

2.1.28.1 Corollary. *Under this equivalence, a derivation from an $\mathbf{F}_1$-algebra A to an A-module M is a map $d\colon A \longrightarrow M$ such that*

1. $d(a+b) = d(a) + d(b)$;
2. $d(ab) = ad(b) + bd(a)$;
3. $$\lambda_n(d(a)) = \sum_{k|n} \lambda_k(a)^{n/k-1} d(\lambda_k(a)).$$

For any $\mathbf{F}_1$-algebra A and any A-module M, a derivation $A \longrightarrow M$ is in particular a derivation $A \otimes_{\mathbf{F}_1} \mathbf{Z} \longrightarrow M \otimes_{\mathbf{F}_1} \mathbf{Z}$; hence we obtain a comparison homomorphism

$$\Omega_{A \otimes_{\mathbf{F}_1} \mathbf{Z}} \longrightarrow \Omega_A \otimes_{\mathbf{F}_1} \mathbf{Z}.$$

2.1.28.2 Corollary. *The comparison homomorphism above is an isomorphism.*

2.1.29. Perhaps even more down to earth is the following description of A-modules over an $\mathbf{F}_1$-algebra A: consider the twisted monoid algebra $A^{\psi}\mathbf{N}^{\times}$; then $\mathbf{Mod}(A)$ is equivalent to the category of left $A^{\psi}\mathbf{N}^{\times}$-modules.

2.1.30 Example. *Let's consider* $\mathbf{F}_1$ *itself. Recall that in* $\mathbf{F}_1$*, the Adams operations* ψ_p *act trivially on* $\mathbf{Z}$*. So an* $\mathbf{F}_1$*-module is a* $\mathbf{Z}\mathbf{N}^{\times}$*-module, i.e., an abelian group* M *along with commuting operations* $\lambda_p\colon M \longrightarrow M$ *for every* $p \in \Pi$*.*

2.1.31 Example. *Suppose* X *a commutative monoid. Recall that* $\mathbf{N}^{\times}$ *acts on* X *via* $\psi_p(x) = x^p$*; we thus form the semidirect product* $X \rtimes_{\psi} \mathbf{N}^{\times}$*. It is easy to see that an* $\mathbf{F}_1 X$*-module is nothing more than an abelian group with an action of* $X \rtimes_{\psi} \mathbf{N}^{\times}$*.*

As a set, $X \rtimes_{\psi} \mathbf{N}^{\times}$ is $X \times \mathbf{N}^{\times}$, but the monoid action is twisted: $(x, m)(y, n) = (x\psi_m(y), mn)$.

In particular, if $X = C_n$*, then we find than an* $\overline{\mathbf{F}_1}$*-module consists of an abelian group* M*, an automorphism* σ *of* M *of order dividing* n*, and commuting operations* $\lambda_p\colon M \longrightarrow M$ *such that* $\lambda_p \circ \sigma = \sigma^p \circ \lambda_p$*.*

Similarly, if $X = \mathbf{Z}$*, then we find that a quasicoherent sheaf on* $\mathbf{G}_{m,\mathbf{F}_1}$ *is an abelian group* M *with an automorphism* σ *and commuting operations* $\lambda_p\colon M \longrightarrow M$ *such that* $\lambda_p \circ \sigma = \sigma^p \circ \lambda_p$*.*

2.1.32 Example. *A module over the Chebyshev line,* $\mathbf{F}_1[x]$ *is a module over* $\mathbf{Z}[x]^{\psi}\mathbf{N}^{\times}$*; i.e., it's an abelian group* M *with an endomorphism* $x\colon M \longrightarrow M$ *and commuting operators* $\lambda_p\colon M \longrightarrow M$ *such that*

$$\lambda_p \circ x = 2T_p(x/2) \circ \lambda_p.$$

We must compute the $\mathbf{F}_{1^n}$-points of various sheaves over $\mathbf{F}_1$.

2.1.33 Example. *If* X *is a monoid and* $S_X \coloneqq \operatorname{Spec}(\mathbf{F}_1 X)$ *then of course*

$$S_X(\mathbf{F}_{1^n}) \cong \operatorname{Hom}_{\mathbf{F}_1}(\mathbf{F}_1 X, \mathbf{F}_1 C_n) \cong \operatorname{Mor}_{\mathbf{Mon}}(X, C_{n,+}).$$

In particular, we find that

$$\mathbf{A}^1_{\mathbf{F}_1,toric}(\mathbf{F}_{1^n}) = C_{n,+}$$

and

$$\mathbf{G}_{m,\mathbf{F}_1}(\mathbf{F}_{1^n}) = C_n.$$

2.1.34 Example. *If* $\mathbf{A}^1_{\mathbf{F}_1}$ *denotes* Spec *of the Chebyshev line, then*

$$\mathbf{A}^1_{\mathbf{F}_1}(\mathbf{F}_{1^n}) = \operatorname{Hom}_{\Psi}(\mathbf{Z}[x], \mathbf{Z}[C_n]),$$

an element of which picks out an element $f \in \mathbf{Z}[t]/(t^n - 1)$ *such that for any* $p \in \Pi$,

$$f(t^p) = 2T_p(f(t)/2) \quad \mathrm{mod}\ (t^n - 1).$$

2.1.35 Definition. Consider the poset Φ of natural numbers ordered by divisibility. A *truncation set* is a sieve $S \subseteq \Phi$ (equivalently, a subfunctor of the terminal presheaf $\Phi \longrightarrow \mathbf{Set}$. If $n \in \mathbf{N}$, then let S/n denote the pullback of S under the multiplication by n map $n\colon \Phi \longrightarrow \Phi$; that is,

$$S/n \coloneqq \{d \in \Phi \mid nd \in S\}.$$

2.2 *Bökstedt–Hesselholt–Madsen computations of* THH

Fix a smooth scheme X over a perfect field k. Recall that

$$\mathbf{TR}^n(X, p) \coloneqq \mathbf{THH}(X) \left\langle p^{n-1} \right\rangle .$$

The cyclotomic structure on **THH** endows $\mathbf{TR}^\bullet_*(X, p)$ with the structure of a p-typical Witt complex.

2.2.1 Theorem (Hesselholt–Madsen)**.** *There is a canonical ring isomorphism*

$$W_\bullet(X) \xrightarrow{\sim} \mathbf{TR}^\bullet_0(X, p)$$

of cyclotomic Green functors.

Consequently, we obtain a unique morphism of p-typical Witt complexes

$$\eta\colon W_\bullet\Omega^*_X \longrightarrow \mathbf{TR}^\bullet_*(X, p).$$

2.2.2 Theorem (Hesselholt)**.** *The map* η *induces an isomorphism*

$$W_\bullet\Omega^k_X \xrightarrow{\sim} \mathbf{TR}^\bullet_k(X, p)$$

for $k \leq 1$.

2.2.3 Theorem (Hesselholt–Madsen)**.** *For each* $n \geq 1$, *the* $W_n(k)$*-module* $\mathbf{TR}^n_2(k, p)$ *is free of rank* 1, *and the canonical graded* $W_n(k)$*-algebra homomorphism*

$$\mathbf{Sym}(\mathbf{TR}^n_2(k, p)) \longrightarrow \mathbf{TR}^n_*(k, p)$$

is an isomorphism.

2.2.4 Theorem (Hesselholt). *For each $n \geq 1$, the canonical graded $W_n(k)$-algebra homomorphism*

$$W_n\Omega^*_X \otimes_{f^*W_n(k)} f^*\mathbf{TR}^n_*(k,p) \longrightarrow \mathbf{TR}^n_*(X,p)$$

is an isomorphism.

2.3 *Regularised products and determinants*

2.3.1 Definition. Suppose, for any $n \in \mathbf{N}$, that λ_n is a complex number with chosen argument α_n. Assume that $N \in \mathbf{N}$ such that $\lambda_n \neq 0$ for any $n \geq N$. Consider the Dirichlet series

$$\sum_{n\geq N} |\lambda_n|^{-s} \exp(-is\alpha_n),$$

which converges on some half plane $\rangle M, +\infty\langle$. Assume also that this sum admits an analytic continuation to a holomorphic function $\zeta_N(s)$ on a half-plane $\rangle -\varepsilon, +\infty\langle$. The the *regularised product*

$$\prod_{n\in\mathbf{N}}(\lambda_n, \alpha_n) := \left(\prod_{n=1}^{N-1}\right)\exp(-\zeta_N'(0)).$$

We say the *regularised product converges* when the assumptions above hold.

2.3.2. We shall always take the branch of the argument lying in $(-\pi, \pi]$. The existence and value of the regularised product is independent of the enumeration, so we are free to write $\prod_{n\in S}\lambda_n$ for a countable set S.

2.3.3 Definition. Suppose V a $\mathbf{C}$-vector space of countable dimension, and suppose that Θ is an endomorphism of V. Assume the following:

- V is the direct sum $\bigoplus_{\lambda\in\mathbf{C}} V_\lambda$, where $V_\lambda = \ker(\Theta - \lambda)^N$ for $N \gg 0$ is finite-dimensional.
- Let (λ_n) be the sequence of eigenvalues of Θ, with multiplicity. The regularised product

$$\prod_n \lambda_n$$

converges.

Then we define the *regularised determinant* as

$$\det_{\infty}(\Theta) := \prod_{n} \lambda_n.$$

2.3.4 Exercise. If $0 \to V' \to V \to V'' \to 0$ is an exact sequence of vector spaces and endomorphisms, then

$$\det_{\infty}(\Theta) = \det_{\infty}(\Theta') \det_{\infty}(\Theta''),$$

in the sense that if the right side converges, then so does the left, and then the values are equal.

2.3.5 Example. *If $\gamma \in \mathbf{C}^{\times}$ and $z \in \mathbf{C}$, then let us compute*

$$\prod_{n \in \mathbf{Z}} \gamma(z+n).$$

The Hurwitz zeta function is defined on $\rangle 1, +\infty \langle$ by

$$\zeta(s,z) := \sum_{n \in \mathbf{N}_0} \frac{1}{(z+n)^s},$$

and it admits an analytic continuation to $\mathbf{C}$ with a pole at 1. We can introduce

$$\zeta_{\gamma}(s,z) := \sum_{n \in \mathbf{N}_0} \frac{1}{(\gamma(z+n))^s},$$

and we can compute that

$$\prod_{n \in \mathbf{Z}} \gamma(z+n) = (z\gamma)^{-1} \exp(-\frac{d}{ds}\zeta_{\gamma}(0,z)) \exp(-\frac{d}{ds}\zeta_{-\gamma}(0,-z)).$$

2.3.6 Proposition. *Let V be an anticommutative graded $\mathbf{C}$-algebra such that $V_j \subset V$ is finite dimensional for all j. Suppose Θ a graded $\mathbf{C}$-linear derivation, and suppose that there exists a unit $\beta \in V_{-2}$ such that*

$$\Theta(v) = \frac{2\pi i}{\log q}\beta.$$

Then

$$\det_{\infty}(s - \Theta | V_{2*+j}) = \det(1 - q^{-s\Theta} | T_j)$$

Now we define the graded derivation Θ on $\mathbf{TP}_*(X)$ for X smooth over $\mathbf{F}_q$ as follows. For the periodicity class $\beta \in \mathbf{TP}_{-2}(X)$, we set

$$\Theta(v) = \frac{2\pi i}{\log q}\beta.$$

and we define Θ on $\mathbf{TP}_0$ and $\mathbf{TP}_1$ so that $q^{\Theta} = \mathbf{Fr}_q^*$. (This can actually be done in more than one manner, but this point isn't important.)

We thus find that

$$\det_{\infty}(s - \Theta|\mathbf{TP}_{2*+j}(X) \otimes_W \mathbf{C}) = \det(1 - q^{-s}\mathbf{Fr}_q^*|\mathbf{TP}_j \otimes_W \mathbf{C})$$

3
Connes–Consani on archimedean L-factors

3.1 Hodge structures and L-factors

3.1.1 Definition. Denote by $\mathbf{S}$ the real algebraic group $\mathbf{C}^\times$; that is, $\mathbf{S}$ is the Weil restriction of the complex algebraic group $\mathbf{G}_m$ to $\mathbf{R} \subset \mathbf{C}$. Denote by $w\colon \mathbf{G}_m \longrightarrow \mathbf{S}$ the canonical morphism that on real points is the inclusion $\mathbf{R}^\times \hookrightarrow \mathbf{C}^\times$.

Now a *Hodge structure* is a finite rank abelian group $\mathrm{H}_{\mathbf{Z}}$ along with an action σ of the real algebraic group $\mathbf{S}$ on $\mathrm{H}_{\mathbf{R}} \coloneqq \mathrm{H}_{\mathbf{Z}} \otimes \mathbf{R}$. We will say that $(\mathrm{H}_{\mathbf{Z}}, \sigma)$ is *pure of weight k* if the action of $\sigma w(t)$ on $\mathrm{H}_{\mathbf{R}}$ is action by t^k for any $t \in \mathbf{R}^\times$.

3.1.2. An action of $\mathbf{S}(\mathbf{C}) \cong \mathbf{C}^\times \times \mathbf{C}^\times$ on $\mathrm{H}_{\mathbf{C}} \coloneqq \mathrm{H}_{\mathbf{Z}} \otimes \mathbf{C}$ is specified by the decomposition

$$\mathrm{H}_{\mathbf{C}} = \bigoplus_{p,q\in\mathbf{Z}} \mathrm{H}^{p,q}, \quad \mathrm{H}^{p,q} = \{x \in \mathrm{H}_{\mathbf{C}} \mid \forall (u,v) \in \mathbf{S}(\mathbf{C}),\ (u,v)x = u^{-p}v^{-q}x\}.$$

This representation is real just in case $\overline{\mathrm{H}^{q,p}} = \mathrm{H}^{p,q}$. Hence a Hodge structure can be defined as such a decomposition. This Hodge structure is of weight k if and only if $\mathrm{H}^{p,q} = 0$ unless $p + q = k$.

This decomposition also specifies a filtration of $\mathrm{H}_{\mathbf{C}}$, called the *Hodge filtration*:

$$\cdots F^{p+1}\mathrm{H} \subset F^p\mathrm{H} \subset \cdots \subset \mathrm{H}_{\mathbf{C}},$$

given by

$$F^p\mathrm{H} \coloneqq \bigoplus_{r\geq p, s\in\mathbf{Z}} \mathrm{H}^{r,s}.$$

The Hodge structure H is pure of weight k just in case $F^q\mathrm{H} \cap \overline{F^p\mathrm{H}} = 0$ whenever $p + q = k + 1$.

3.1.3 Example. *Define a Hodge structure*

$$\mathbf{Z}(1) \coloneqq (2\pi\sqrt{-1})\mathbf{Z} = \ker\left[\exp\colon \mathbf{C} \longrightarrow \mathbf{C}^\times\right] \quad \textit{with} \quad \mathbf{Z}(1)^{-1,-1} = \mathbf{Z}(1);$$

this is called the Tate Hodge structure, pure of weight −2. Its tensor powers are Hodge structures

$$\mathbf{Z}(n) \coloneqq (2\pi\sqrt{-1})^n\mathbf{Z} \subset \mathbf{C} \quad \textit{with} \quad \mathbf{Z}(n)^{-n,-n} = \mathbf{Z}(n);$$

these are pure of weight $-2n$.

3.1.4 Example. *If X is a compact Kähler manifold, then the holomorphic Poincaré lemma guarantees a quasi-isomorphism $\Omega_X \coloneqq \Omega_X^\bullet \simeq \mathbf{C}_X$. Now the "foolish" filtration*

$$\cdots \longrightarrow \Omega_X^{\geq n} \longrightarrow \Omega_X^{\geq n-1} \longrightarrow \cdots \longrightarrow \Omega_X$$

gives rise to a spectral sequence

$$E_1^{p,q} = \mathrm{H}^q(X, \Omega_X^p) \Longrightarrow \mathrm{H}^{p+q}(X, \mathbf{C})$$

whose abutment is the Hodge filtration on $\mathrm{H}^*(X, \mathbf{C})$. *From Hodge theory, we know that this spectral sequence degenerates, whence we obtain a decomposition*

$$\mathrm{H}^k(X, \mathbf{C}) = \bigoplus_{p+q=k} \mathrm{H}^q(X, \Omega^p).$$

Moreover, one has $\overline{\mathrm{H}^p(X, \Omega^q)} = \mathrm{H}^q(X, \Omega^p)$. *Thus the singular cohomology* $\mathrm{H}^k(X, \mathbf{Z})$ *is a Hodge structure pure of weight k.*

This is all neatly summarized in the statement that the singular cohomology of compact Kähler manifolds (and thus of smooth projective varieties over **C**) *is "really" valued in the category of Hodge structures.*

3.1.5 Notation. Recall

$$\begin{aligned} \Gamma_{\mathbf{R}}(s) &\coloneqq \pi^{-s/2}\Gamma(s/2) \\ \Gamma_{\mathbf{C}}(s) &\coloneqq 2(2\pi)^{-s}\Gamma(s), \end{aligned}$$

so that

$$\Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1).$$

If $\mathrm{H}_{\mathbf{C}}$ is a complex representation of $\mathbf{C}^{\times} \times \mathbf{C}^{\times}$ (i.e., a $\mathbf{C}$-Hodge structure), then we write $h(p,q) = \dim_{\mathbf{C}} \mathrm{H}^{p,q}$, and

$$\Gamma_{\mathrm{H}_{\mathbf{C}}} := \prod_{p,q} \Gamma_{\mathbf{C}}(s - \min\{p,q\})^{h(p,q)}.$$

If $\mathrm{H}_{\mathbf{R}}$ is a Hodge structure, then one defines

$$\begin{aligned} \mathrm{H}^{n,+} &:= \{x \in \mathrm{H}^{n,n} \mid \overline{x} = (-1)^n x\}; \\ \mathrm{H}^{n,-} &:= \{x \in \mathrm{H}^{n,n} \mid \overline{x} = -(-1)^n x\} \end{aligned}$$

and

$$\begin{aligned} h(n,+) &:= \dim_{\mathbf{C}} \mathrm{H}^{n,+}; \\ h(n,-) &:= \dim_{\mathbf{C}} \mathrm{H}^{n,-}. \end{aligned}$$

Now

$$\Gamma_{\mathrm{H}_{\mathbf{R}}} := \prod_{n} \Gamma_{\mathbf{R}}(s-n)^{h(n,+)} \Gamma_{\mathbf{R}}(s-n+1)^{h(n,-)} \prod_{p<q} \Gamma_{\mathbf{C}}(s-p)^{h(p,q)}.$$

In particular, suppose X a smooth projective variety over a global field K, and let v be an archimedean place of K. If $K_v \cong \mathbf{C}$, then $\mathrm{H}^w(X(K_v)^{an}, \mathbf{C})$ admits a $\mathbf{C}$-Hodge structure, and we write

$$L_v(h^w(X), s) := \Gamma_{\mathrm{H}^w(X(K_v)^{an}, \mathbf{C})}(s).$$

If $K_v \cong \mathbf{R}$, then $\mathrm{H}^w(X(K_v(i))^{an}, \mathbf{C})$ admits a Hodge structure, and we write

$$L_v(h^w(X), s) := \Gamma_{\mathrm{H}^w(X(K_v(i))^{an}, \mathbf{C})}(s).$$

3.2 *Deligne cohomology and Beilinson's theorem*

Deligne cohomology can be thought of as a systematic way of packaging both ordinary cohomology and the intermediate Jacobians.

3.2.1 Definition. For any integer p, the *Deligne complex* $\mathbf{Z}(p)_{\mathrm{D}}$ is a sheaf of complexes (or, better, simplicial abelian groups, or, still better, spectra) on the site of complex analytic manifolds defined as the homotopy fiber product:

$$\mathbf{Z}(p)_{\mathrm{D}} := \mathbf{Z}(p) \times^h_{\mathbf{C}} \Omega^{\geq p}.$$

Its cohomology groups on a compact complex analytic manifold X are the *Deligne cohomology groups*:

$$\mathrm{H}^q_{\mathrm{D}}(X, \mathbf{Z}(p)) := \mathrm{H}^q(X, \mathbf{Z}(p)_{\mathrm{D}}).$$

3.2.2. Equivalently, $\mathbf{Z}(p)_{\mathrm{D}}$ is the fiber of the natural map $\mathbf{Z}(p) \longrightarrow \mathbf{C}/F^p\Omega$.

One can form an explicit complex of sheaves of abelian groups on a complex analytic manifold that represents $\mathbf{Z}(p)_{\mathrm{D}}$:

$$0 \to \mathbf{Z}_X(p) \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{p-1}_X \to 0.$$

3.2.3 Example. *Of course* $\mathbf{Z}(0)_D = \mathbf{Z}$, *and so*

$$\mathrm{H}^q_D(X, \mathbf{Z}(0)) \cong \mathrm{H}^q(X, \mathbf{Z}).$$

3.2.4 Example. *The map*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{Z}(1) & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} & & \downarrow \\ 1 & \longrightarrow & 1 & \longrightarrow & \mathcal{O}_X^\times & \longrightarrow & 1 \end{array}$$

is an equivalence $\exp\colon \mathbf{Z}_X(1)_D \xrightarrow{\sim} \mathcal{O}_X^\times[-1]$, *and so*

$$\mathrm{H}^q_D(X, \mathbf{Z}(1)) \cong \mathrm{H}^{q-1}(X, \mathcal{O}_X^\times).$$

In particular, note that $\mathrm{H}^2_D(X, \mathbf{Z}(1)) \cong \mathrm{Pic}(X)$, *and we have an exact sequence*

$$0 \to J^1(X) \to \mathrm{H}^2_D(X, \mathbf{Z}(1)) \to \mathbf{NS}(X) \to 0.$$

3.2.5 Example. *There is an equivalence*

$$\mathbf{Z}_X(2)_D \simeq \left[d\log\colon \mathcal{O}_X^\times \longrightarrow \Omega^1_X\right][-1].$$

One can show that $\mathrm{H}^2_D(X, \mathbf{Z}(2))$ *is the group of line bundles with holomorphic connection.*

3.2.6. The long exact sequence of the fiber gives

$$\mathrm{H}^{q-1}(X, \mathbf{Z}(p)) \to \frac{\mathrm{H}^{q-1}(X, \mathbf{C})}{F^p\mathrm{H}^{q-1}(X, \mathbf{C})} \to \mathrm{H}^q_{\mathrm{D}}(X, \mathbf{Z}(p)) \to \mathrm{H}^q(X, \mathbf{Z}(p)) \to \frac{\mathrm{H}^q(X, \mathbf{C})}{F^p\mathrm{H}^q(X, \mathbf{C})}.$$

When $q < 2p$, the map on the right is an injection, so we have a short exact sequence

$$0 \to J^p\mathrm{H}^{q-1}(X, \mathbf{Z}(p)) \to \mathrm{H}^q_{\mathrm{D}}(X, \mathbf{Z}(p)) \to \mathrm{H}^q(X, \mathbf{Z}(p)) \to 0.$$

When $q = 2p$, we get a short exact sequence

$$0 \to J^p(X) \to \mathrm{H}^{2p}_{\mathrm{D}}(X, \mathbf{Z}(p)) \to \mathbf{Hdg}^p(X) \to 0.$$

If X is a smooth projective variety over a global field K, then for any archimedean place v of K, we write

$$\mathrm{H}^q_{\mathrm{D}}(X_v, \mathbf{R}(p)) := \begin{cases} \mathrm{H}^q_{\mathrm{D}}(X_v, \mathbf{R}(p)) & \text{if } v \text{ is complex;} \\ \mathrm{H}^q_{\mathrm{D}}(X_v \otimes_{K_v} K_v(i), \mathbf{R}(p))^{C_2} & \text{if } v \text{ is real,} \end{cases}$$

where C_2 acts via de Rham conjugation.

3.2.7 Theorem (Beilinson)**.**

$$\mathrm{ord}_{s=m} L_v(\mathrm{H}^w(X), s)^{-1} = \dim_{\mathbf{R}} \mathrm{H}^{w+1}_D(X_v, \mathbf{R}(w+1-m)).$$

3.2.8 Definition. Suppose $X_{\mathbf{C}}$ a smooth complex projective variety. Consider the Fréchet algebras $C^\infty(X^{an}_{\mathbf{C}}, \mathbf{R})$ and $C^\infty(X^{an}_{\mathbf{C}}, \mathbf{C})$; using a topologised version of the Hochschild complex, $\mathbf{TH}^{cts} \otimes \mathbf{C}$, we obtain an identification

$$\mathbf{TP}(X_{\mathbf{C}}) \otimes \mathbf{C} \simeq \mathbf{TP}^{cts}(C^\infty(X^{an}_{\mathbf{C}}, \mathbf{C})) \otimes \mathbf{C},$$

and we define

$$\mathbf{TP}_{\mathbf{R}}(X_{\mathbf{C}}) \otimes \mathbf{C} := \mathbf{TP}^{cts}(C^\infty(X^{an}_{\mathbf{C}}, \mathbf{R})) \otimes \mathbf{C}.$$

Now we have a natural λ-algebra structure on all these invariants, and we define Θ_0 as the generator of the λ operations, so that

$$k^{\Theta_0} = \lambda_k.$$

Now we select the map

$$(2\pi i)^{\Theta_0} : \mathbf{TP}_{\mathbf{R}}(X_{\mathbf{C}}) \otimes \mathbf{C} \longrightarrow \mathbf{TP}(X_{\mathbf{C}}) \otimes \mathbf{C}.$$

Recall that we have the map $\mathbf{TC}(X_{\mathbf{C}}) \longrightarrow \mathbf{TP}(X_{\mathbf{C}})$. We form the homotopy pullback

$$\mathbf{P}^{an}(X_{\mathbf{C}}) := \Big((\mathbf{TC}(X_{\mathbf{C}}) \otimes \mathbf{C}) \times^h_{\mathbf{TP}(X_{\mathbf{C}}) \otimes \mathbf{C},\ (2\pi i)^{\Theta_0}} \mathbf{TP}_{\mathbf{R}}(X_{\mathbf{C}}) \otimes \mathbf{C} \Big) [1].$$

For a smooth real projective variety $X_{\mathbf{R}}$, define

$$\mathbf{P}^{an}(X_{\mathbf{R}}) := \mathbf{P}^{an}(X_{\mathbf{C}})^{C_2},$$

where C_2 acts on the coefficients.

There are two domains of interest:

$$E_d = \{(n, j) \mid n \geq 0,\ 0 \leq 2j - n \leq 2d\}$$

and

$$A_d = \{(q, m) \mid 0 \leq q \leq 2d,\ m \leq q/2\}$$

3.2.9 Proposition. *There are isomorphisms*

$$\pi_n(\mathbf{P}^{an}(X_{\mathbf{C}}))^{\Theta_0 = j} \cong \mathrm{H}_D^{2j+1-n}(X_{\mathbf{C}}, \mathbf{R}(j+1))$$

and

$$\pi_n(\mathbf{P}^{an}(X_{\mathbf{R}}))^{\Theta_0 = j} \cong \mathrm{H}_D^{2j+1-n}(X_{\mathbf{C}}, \mathbf{R}(j+1))$$

for $(n, j) \in E_d$ *(else the left side vanishes).*