On Galois categories & perfectly reduced schemes

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Let X be a scheme.[1] A point $x = \operatorname{Spec} \kappa(x) \to X$ of X has an image Zariski point $x_0 \in X^{zar}$ with residue field $\kappa(x_0) \subseteq \kappa(x)$. Let us say that x is a *geometric* point when $\kappa(x)$ is a separable closure of $\kappa(x_0)$. Geometric points constitute a category,[2] which we call the *Galois category* $\operatorname{Gal}(X)$. The morphisms $x \to y$ are *specialisations* $x \leftsquigarrow y$ – i.e., natural transformations between the corresponding morphisms of topoi $x_* \leftarrow y_*$ (or, if you prefer, $x^* \to y^*$). In other words, $\operatorname{Gal}(X)$ is the category of points of the étale topos of X. It is a 1-category in which every endomorphism is an automorphism. It comes equipped with a profinite topology; that is, $\operatorname{Gal}(X)$ is a category object in Stone topological spaces.

The Galois category also comes equipped with a conservative functor $\operatorname{Gal}(X) \to X^{zar}$, whose target is a poset under specialisation; this functor is continuous for the profinite topologies.[3] Accordingly, X^{zar} is the poset of isomorphism classes of objects of $\operatorname{Gal}(X)$.

The profinite category $\operatorname{Gal}(X)$ is determined by the étale topos of X, but it also determines it; in fact, if you're a hyperpolyglot, you can probably deduce this already from Makkai's Strong Conceptual Completeness Theorem.[4] We took[5] an explicit approach that showed that étale sheaves on X 'are' continuous representations $\operatorname{Gal}(X)$, generalising the usual equivalence between the étale cohomology and the Galois cohomology of a field.

If $X^{zar} \to P$ is a finite constructible stratification of a scheme, then the *Galois ∞-category* $\operatorname{Gal}(X/P)$ is what you get by localising (in the wholesome ∞-categorical sense) the specialisations that occur within any single stratum. The result is a profinite ∞-category with a conservative functor to P – what we called a *profinite P-stratified space*.[6] It is the ∞-category of points of the *P-stratified ∞-topos*. In the extreme case, when P is the trivial poset, $\operatorname{Gal}(X/*)$ is the profinite étale homotopy type. Hence $\operatorname{Gal}(X)$ is a complete delocalisation of the étale homotopy type.

When you view $\operatorname{Gal}(X)$ through this lens, you get to interpret it as a profinite stratified space whose underlying space is the profinite étale homotopy type $\operatorname{Gal}(X/*)$. Each irreducible closed subscheme $Z \subseteq X$ identifies the closure $[Z]$ of a stratum within X. If $Z \subseteq W$ are two irreducible closed subschemes of X, then the space of sections of $\operatorname{Gal}(X) \to X^{zar}$ over the edge $\eta_Z \to \eta_W$ of the generic points is the deleted tubular neighbourhood[7] of $[Z]$ in $[W]$. This stratified space is a stratified 1-type: the strata and deleted tubular neighbourhoods are all $K(\pi, 1)$'s.

Example (Fields)**.** If k is a field, then a choice of a separable closure of k identifies an equivalence $\operatorname{Gal}(\operatorname{Spec} k) \simeq BG_k$, where G_k is the absolute Galois group of k.

Example (Knots and primes)**.** If A is a number ring with fraction field K, then $\operatorname{Gal}(\operatorname{Spec} A)$ is a category with (isomorphism classes of) objects the prime ideals of A. For each nonzero prime ideal $p \in \operatorname{Spec} A$, the automorphisms of p can be identified with the absolute Galois group $G_{\kappa(p)}$

[1] We only work with coherent schemes, which out of indolence we just call *schemes*.

[2] *Théorie des topos et cohomologie étale des schémas. Tome 2.* Séminaire de Géométrie Algébrique du Bois Marie 1963–64 (SGA 4). Dirigé par M. Artin, A. Grothendieck, J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint–Donat. Lecture Notes in Mathematics, Vol. 270. Berlin: Springer-Verlag, 1963–64, pp. iv+418 (henceforth cited as SGA 4II), Exposé VIII, §7.

[3] The topological space X^{zar} is a *spectral topological space*, which is the same thing as a profinite poset.

[4] M. Makkai. *Stone duality for first order logic.* In: Adv. in Math. 65.2 (1987), pp. 97–170. DOI: 10.1016/0001-8708(87)90020-X.

[5] C. Barwick, S. Glasman, and P. Haine. *Exodromy.* Preprint arXiv:1807.03281. July 2018.

[6] Barwick, Glasman, and Haine.

[7] This is literally the étale homotopy type of the *oriented fibre product* $\eta_Z \vec{\times}_X \eta_W$.

of the finite field $\kappa(p)$. Thus the étale homotopy type of Spec A is stratified by the various closed strata, each of which is an embedded circle – i.e., a knot $BG_{\kappa(p)}$. The open complement of each $BG_{\kappa(p)}$ is a BG_p, where $G_p := \pi_1(\text{Spec } A \smallsetminus p)$ is the automorphism group of the maximal Galois extension of K that is ramified at most only at p and the infinite primes. Enveloping each knot is a tubular neighbourhood, given by Gal(Spec A_p^{sh}) (sh=strict henselisation), so that the deleted tubular neighbourhood of $BG_{\kappa(p)}$ is a BG_{K_p}.

Example (Analytification)**.** If X is a finite type F-scheme, where F is $\boldsymbol{C}$, $\boldsymbol{R}$, or any nonarchimedean field, then there is an associated X^{an} analytic space, which admits a profinite stratification by X^{zar}. The category Gal(X) is the profinite completion of the exit-path ∞-category of X^{an} with this stratification. (We proved this over $\boldsymbol{C}$,[8] but the same proof will work any time you have access to an Artin Comparison Theorem, which you do in these situations; see, e.g., Berkovich's proof.[9])

[8] Barwick, Glasman, and Haine, Proposition 13.15 & Corollary 13.16.

[9] V. G. Berkovich. *On the comparison theorem for étale cohomology of non-Archimedean analytic spaces.* In: Israel J. Math. 92.1-3 (1995), pp. 45–59. DOI: 10.1007/BF02762070.

The *perfectly reduced schemes* of the title are schemes taken up to universal homeomorphism (Definition 4.2). Grothendieck's *invariance topologique* of the étale topos[10] ensures that the *only* kinds of schemes Galois categories can hope to capture in their entirety are the perfectly reduced schemes. This note[11] is the suggestion of a recognition principle that flows in the opposite direction; that is, we aim to read off facts about perfectly reduced schemes from their Galois categories. Our goal is a dictionary between the geometric features of a perfectly reduced scheme (or morphism of such) and the categorical properties of its Galois category (or functor of such); the gnomic section titles are the first few entries in this dictionary. The main new thing that makes these entries possible is the total separable closure of Stefan Schröer.[12]

[10] SGA 4II, Exposé VIII, 1.1.

[11] I thank Peter Haine for sharing his many insights about this material. I am also grateful to the Isaac Newton Institute in Cambridge, whose hospitality I enjoyed as I completed this work.

[12] S. Schröer. *Geometry on totally separably closed schemes.* In: Algebra Number Theory 11.3 (2017), pp. 537–582. DOI: 10.2140/ant.2017.11.537.

Contents

1 *Open = cosieve & closed = sieve*

Let us begin with the obvious.

1.1 Proposition. *A monomorphism* $U \hookrightarrow X$ *of schemes is an open immersion if and only if the induced functor* $\mathrm{Gal}(U) \to \mathrm{Gal}(X)$ *is equivalent to the inclusion of a cosieve.*

Dually, a monomorphism $Z \hookrightarrow X$ *of schemes is a closed immersion if and only if* $\mathrm{Gal}(Z) \to \mathrm{Gal}(X)$ *is equivalent to the inclusion of a sieve.*

An *interval* in an ∞-category C is a full subcategory $D \subseteq C$ such that a morphism $P \to Q$ of D factors through an object R of C only if R lies in D.

1.2 Corollary. *A monomorphism* $W \hookrightarrow X$ *of schemes is a locally closed immersion if and only if the induced functor* $\mathrm{Gal}(W) \to \mathrm{Gal}(X)$ *is equivalent to the inclusion of an interval.*

1.3 Corollary. *A scheme* X *is local if and only if* $\mathrm{Gal}(X)$ *contains a weakly initial object – i.e., an object from which every object receives a morphism. Dually, a scheme* X *is irreducible if and only if* $\mathrm{Gal}(X)$ *contains a weakly terminal object – i.e., an object to which every object sends a morphism.*

1.4. For any scheme X and any point $x_0 \in X^{zar}$, the Galois category of the localisation is the fibre product

$$\mathrm{Gal}(X_{(x_0)}) \simeq \mathrm{Gal}(X) \times_{X^{zar}} X^{zar}_{x/} \, .$$

Dually, for any point $y_0 \in X^{zar}$, the Galois category of the closure $X^{(y_0)}$ of y_0 (with the reduced subscheme structure, say) is the fibre product

$$\mathrm{Gal}(X^{(y_0)}) \simeq \mathrm{Gal}(X) \times_{X^{zar}} X^{zar}_{/y} \, .$$

2 *Strict localisation = undercategory & strict normalisation = overcategory*

2.1 Notation. If $x \to X$ is a point of a scheme X, then we write O^h_{X,x_0} for the henselisation of the local ring O_{X,x_0}, and we write $O^h_{X,x} \supseteq O^h_{X,x_0}$ for the unique extension of henselian local rings that on residue fields reduces to the field extension $\kappa \supseteq \kappa(x_0)$, where κ is the separable closure of $\kappa(x_0)$ in $\kappa(x)$. We will also write

$$X_{(x)} \coloneqq \operatorname{Spec} O^h_{X,x} \, .$$

We call $X_{(x)}$ the *localisation of* X *at* x. It is the limit of the factorisations $x \to U \to X$ in which $U \to X$ is étale.

If $x \to X$ is a geometric point, then $O^h_{X,x}$ is the strict henselisation of O_{X,x_0}, and $X_{(x)}$ is the strict localisation of X at x.

Dually, if $y \to X$ is a point, then we write $X^{(y_0)}$ for the reduced subscheme structure on the Zariski closure of y_0, and we write $X^{(y)}$ for the normalisation of $X^{(y_0)}$ under $\operatorname{Spec} \kappa$, where κ is the separable closure of $\kappa(y_0)$ in $\kappa(y)$. We call $X^{(y)}$ the *normalisation of* X *at* y.

If $y \to X$ is a geometric point, then we call $X^{(y)}$ the *strict normalisation of* X *at* y. It is the limit of the factorisations $y \to Z \to X$ in which $Z \to X$ is finite.

2.2. Stefan Schröer[13] has brought us *totally separably closed* schemes, which

[13] Schröer.

are integral normal schemes whose function field is separably closed. In other words, a totally separably closed scheme is one of the form $X^{(y)}$ for some geometric point $y \to X$. (In the language of Schröer, $X^{(y)}$ is the total separable closure of the Zariski closure of y_0 – with the reduced subscheme structure – under y.) Schröer has shown that this class of schemes has a number of curious properties:

- If Z is totally separably closed, then for any point $z_0 \in Z^{zar}$, the local ring O_{Z,z_0} is strictly henselian.[14]
- If Z is totally separably closed, then the étale topos and the Zariski topos of Z coincide, so that $\mathrm{Gal}(Z) \simeq Z^{zar}$.[15] In other words, $\mathrm{Gal}(Z)$ is a profinite poset with a terminal object.
- If Z is totally separably closed and W is irreducible, then any integral morphism $W \to Z$ is radicial.[16] Thus any integral surjection $W \to Z$ is a universal homeomorphism.
- If Z is totally separably closed, then the poset $\mathrm{Gal}(Z) \simeq Z^{zar}$ has all finite nonempty joins.[17]

[14] Schröer, Proposition 2.6.

[15] Schröer, Corollary 2.5.

[16] Schröer, Lemma 2.3.

[17] S. Schröer. *Total separable closure and contractions*. Preprint arXiv:1708.06593. Aug. 2017, Theorem 2.1.

Here now is the basic observation, which follows more or less immediately from the limit descriptions of the strict localisation and the strict normalisation:

2.3 Proposition. *Let X be a scheme, and let $x \to X$ and $y \to X$ be two geometric points thereof. The following profinite sets are in (canonical) bijection:*

- *the set* $\mathrm{Map}_{\mathrm{Gal}(X)}(x, y)$ *of morphisms $x \to y$ in* $\mathrm{Gal}(X)$*;*
- *the set* $\mathrm{Mor}_X(y, X_{(x)})$ *of lifts of y to the strict localisation $X_{(x)}$;*
- *the set* $\mathrm{Mor}_X(x, X^{(y)})$ *of lifts of y to the strict normalisation $X^{(y)}$.*

We may thus describe the over- and undercategories of Galois categories:

2.4 Corollary. *Let X be a scheme, and let $x \to X$ and $y \to X$ be two geometric points thereof. Then we have*

$$\mathrm{Gal}(X)_{x/} \simeq \mathrm{Gal}(X_{(x)}) \quad \textit{and} \quad \mathrm{Gal}(X)_{/y} \simeq \mathrm{Gal}(X^{(y)})\,.$$

The first sentence is originally due to Grothendieck.[18]

[18] SGA 4II, Exposé VIII, Corollaire 7.6.

2.5 Corollary. *Let X be a scheme. Then* $\mathrm{Gal}(X)$ *is equivalent to both of the following full subcategories of X-schemes:*

- *the one spanned by the strict localisations of X, and*
- *the one spanned by the strict normalisations of X.*

Since $\mathrm{Gal}(X^{(y)}) \simeq X^{(y),zar}$, it follows that Galois categories are of a very particular sort:

2.6 Corollary. *Let X be a scheme. For any geometric point $y \to X$, the overcategory $\mathrm{Gal}(X)_{/y}$ is a profinite poset with all finite nonempty joins. In particular, every morphism of $\mathrm{Gal}(X)$ is a monomorphism.*

2.7 Definition. Let X be a scheme. Then a *witness* is a totally separably closed valuation ring V and a morphism $\gamma\colon \operatorname{Spec} V \to X$. If p_0 is the initial object of $\mathrm{Gal}(V)$ and p_∞ is the terminal object of $\mathrm{Gal}(V)$, then we say that γ *witnesses* the map $\gamma(p_0) \to \gamma(p_\infty)$ of $\mathrm{Gal}(X)$.

2.8. Any morphism $x \to y$ of $\mathrm{Gal}(X)$ has a witness: you can always find a local morphism $\operatorname{Spec} V \to (X^{(y)})_{(x)}$ that induces an isomorphism of function fields.

3 *Universal homeomorphism = equivalence*

Now we arrive at a sensitive question: under which circumstances does a morphism of schemes induce an equivalence of étale topoi or, equivalently, of Galois categories? The well-known theorem here is Grothendieck's *invariance topologique* of the étale topos,[19] which states that a universal homeomorphism induces an equivalence on étale topoi. Let us reprove this result with the aid of Galois categories; this will also provide us with a partial converse.

[19] SGA 4II, Exposé VIII, 1.1.

3.1 Proposition. *Let $f: X \to Y$ be a morphism of schemes. If f is radicial, then every fibre of $\mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ is either empty or a singleton.*[20] *Conversely, if f is of finite type, and if every fibre of $\mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ is either empty or a singleton, then f is radicial.*

[20] By *singleton* we mean *contractible groupoid*.

Proof. If f is radicial, then the map $X^{zar} \to Y^{zar}$ is an injection, and for any point $x_0 \in X^{zar}$, the map $BG_{\kappa(x_0)} \to BG_{\kappa(f(x_0))}$ on fibres is an equivalence since $\kappa(f(x_0)) \subseteq \kappa(x_0)$ is purely inseparable. So for any geometric point y with image y_0, the fibre over y is a singleton.

Conversely, if f is of finite type, and if every fibre of $\mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ is either empty or a singleton, then certainly the map $X^{zar} \to Y^{zar}$ is an injection, whence f is in particular quasifinite. For any point $x_0 \in X^{zar}$, the fibres of the map $BG_{\kappa(x_0)} \to BG_{\kappa(f(x_0))}$ are each a singleton, whence it is an equivalence. Now since $\kappa(f(x_0)) \subseteq \kappa(x_0)$ is a finite extension, it is purely inseparable. □

3.2 Example. The finite type hypothesis in the second half of Proposition 3.1 is of course necessary, as any nontrivial extension $E \subset F$ of separably closed fields induces the identity on trivial Galois categories.

3.3 Corollary. *Let $f: X \to Y$ be a morphism of schemes. If f is radicial and surjective, then every fibre of $\mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ is a singleton. Conversely, if f is of finite type, and if every fibre of $\mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ is a singleton, then f is radicial and surjective.*

The following is the Valuative Criterion, along with a simple argument[21]

[21] The Stacks Project Authors. *Stacks Project*. stacks.math.columbia.edu. 2018 (henceforth cited as STK), Tag 03K8.

that allows one to extend the fraction field of the valuation ring therein.

3.4 Lemma. *Let $f\colon X \to Y$ be a morphism of schemes. Then the following are equivalent.*

- *The morphism f is universally closed.*
- *For any witness $\gamma\colon \operatorname{Spec} V \to Y$ and any diagram*

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ \operatorname{Spec} V & \underset{\gamma}{\longrightarrow} & Y \end{array}$$

  in which K is the fraction field of V, there exists a lift $\overline{\gamma}\colon \operatorname{Spec} V \to X$.

3.5 Recollection. A functor $f\colon C \to D$ is said to be a *right fibration* if and only if, for any object $x \in C$, the induced functor $C_{/x} \to D_{/f(x)}$ is an equivalence of categories. In this case, one may say that C is a *category fibred in groupoids over D*. For any such right fibration, there is a diagram F of groupoids indexed on D^{op} such that C is the Grothendieck construction of F.

Dually, f is a *left fibration* if and only if f^{op} is a right fibration, so that for any object $x \in C$, the induced functor $C_{x/} \to D_{f(x)/}$ is an equivalence of categories.

3.6 Proposition. *Let $f\colon X \to Y$ be a morphism of schemes. If f is an integral morphism, then* $\operatorname{Gal}(X) \to \operatorname{Gal}(Y)$ *is a right fibration. Conversely, if* $\operatorname{Gal}(X) \to \operatorname{Gal}(Y)$ *is a right fibration, then f is universally closed.*

Proof. Assume that f is integral. Then for every geometric point $x \to X$, the induced morphism $X^{(x)} \to Y^{(f(x))}$ is also integral, and by Schröer's result,[22] it is radicial as well. Hence at the level of Zariski topological spaces, $X^{(x),zar} \to Y^{(f(x)),zar}$ is an inclusion of a closed subset; since source and target are each irreducible, and the inclusion carries the generic point to the generic point, it is a homeomorphism. (In fact, $X^{(x)} \to Y^{(f(x))}$ is a universal homeomorphism.) Thus

[22] Schröer, Lemma 2.3.

$$\operatorname{Gal}(X)_{/x} \simeq \operatorname{Gal}(X^{(x)}) \simeq X^{(x),zar} \to Y^{(f(x)),zar} \simeq \operatorname{Gal}(Y^{(f(x))}) \simeq \operatorname{Gal}(Y)_{/f(x)}$$

is an equivalence, whence $\operatorname{Gal}(X) \to \operatorname{Gal}(Y)$ is a right fibration.

Conversely, assume that f is of finite type and that $\operatorname{Gal}(X) \to \operatorname{Gal}(Y)$ is a right fibration. We employ Lemma 3.4 to show that f is universally closed; consider a witness $\gamma\colon \operatorname{Spec} V \to Y$ along with a diagram

$$\begin{array}{ccc} \operatorname{Spec} K & \overset{\xi}{\longrightarrow} & X \\ \big\downarrow & & \big\downarrow{\scriptstyle f} \\ \operatorname{Spec} V & \underset{\gamma}{\longrightarrow} & Y \end{array}$$

in which K is the fraction field of V. Let $\psi\colon y \to f(\xi)$ be the morphism of $\operatorname{Gal}(Y)$ witnessed by γ, and let $\phi\colon x \to \xi$ be a lift thereof to $\operatorname{Gal}(X)$. We obtain a square

$$\begin{array}{ccc} O^{sh}_{Y,y} & \xrightarrow{\gamma} & V \\ \downarrow & & \downarrow \\ O^{sh}_{X,x} & \xrightarrow[\xi]{} & K\,, \end{array}$$

and since $O^{sh}_{Y,y} \to O^{sh}_{X,x}$ is local, we obtain a lift $\overline{\gamma}\colon O^{sh}_{X,x} \to V$, as required. □

A universal homeomorphism is a morphism that is radicial, surjective, and universally closed. An equivalence of categories is a right fibration with fibres contractible groupoids. We thus deduce:

3.7 Proposition. *Let $f\colon X \to Y$ be a morphism of schemes. If f is a universal homeomorphism, then $\mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ is an equivalence. Conversely, if f is of finite type, and if $\mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ is an equivalence, then f is a universal homeomorphism (which is necessarily finite).*

4 *Interlude: perfectly reduced schemes*

A reduced scheme receives no nontrivial nilimmersions; a *perfectly reduced* scheme receives no nontrivial universal homeomorphisms. This is in fact a local condition that can be expressed in very concrete terms:

4.1 Proposition. *The following are equivalent for a scheme X.*

- *There exists an affine open covering $\{\mathrm{Spec}\, A_i\}_{i\in I}$ of X such that for every $i \in I$, the following conditions obtain:*
  - *for any $f, g \in A_i$, if $f^2 = g^3$, then there is a unique $h \in A_i$ such that $f = h^3$ and $g = h^2$; and*
  - *for any prime number p and any $f, g \in A_i$, if $f^p = p^p g$, then there is a unique element $h \in A_i$ such that $f = ph$ and $g = h^p$.*
- *If X' is a reduced scheme and $f\colon X' \to X$ is a universal homeomorphism, then f is an isomorphism.*

4.2 Definition. A scheme that enjoys one and therefore both of the conditions of Proposition 4.1 is said to be *perfectly reduced* or – in the parlance of David Rydh[23] and the Stacks Project[24] – *absolutely weakly normal.*

Let us write $\mathbf{Sch}_{perf} \subset \mathbf{Sch}_{coh}$ for the full subcategory of schemes spanned by the perfectly reduced schemes.

[23] D. Rydh. *Submersions and effective descent of étale morphisms.* In: Bull. Soc. Math. France 138.2 (2010), pp. 181–230, Appendix B.

[24] STK, Tag 0EUL.

4.3. To express this differently, let us define a family of reference universal homeomorphisms. First, let Υ denote the cuspidal cubic

$$\Upsilon \coloneqq \mathrm{Spec}\,\mathbf{Z}[u,v]/(u^2 - v^3)\,.$$

The normalisation $\rho\colon \mathbf{A}^1_{\mathbf{Z}} \to \Upsilon$ defined by the equations $u = t^3$ and $v = t^2$ is a universal homeomorphism. Next, for any prime number p, set

$$Z_p \coloneqq \mathrm{Spec}\,\mathbf{Z}[y,z]/(y^p - p^p z)\,.$$

The normalisation $\tau_p\colon \mathbf{A}^1_{\mathbf{Z}} \to Z_p$ defined by the equations $y = px$ and $z = x^p$ is a universal homeomorphism. Proposition 4.1 states that a scheme X is perfectly reduced if and only if every point $x \in X$ is contained in a Zariski open neighbourhood $U \subseteq X$ such that the map

$$\mathrm{Mor}(U, \mathbf{A}^1_{\mathbf{Z}}) \to \mathrm{Mor}(U, Y)$$

is a bijection, and for any prime number p, the map

$$\mathrm{Mor}(U, \mathbf{A}^1_{\mathbf{Z}}) \to \mathrm{Mor}(U, Z_p)$$

is a bijection.

4.4. Any (quasicompact) open subscheme of a perfectly reduced scheme is perfectly reduced. A reduced $\mathbf{Q}$-scheme is perfectly reduced if and only if it is *seminormal*. A reduced $\mathbf{F}_p$-scheme is perfectly reduced if and only if the Frobenius morphism is an isomorphism.

4.5 Proposition. [25] *The inclusion* $\mathbf{Sch}_{perf} \hookrightarrow \mathbf{Sch}_{coh}$ *admits a right adjoint* $X \mapsto X_{perf}$, *which exhibits* $\mathbf{Sch}_{perf}$ *as the colocalisation of* $\mathbf{Sch}_{coh}$ *along the class of universal homeomorphisms. In particular, the counit* $X_{perf} \to X$ *is the initial object in the category of universal homeomorphisms to* X. *We call* X_{perf} *the perfection of* X.

[25] Barwick, Glasman, and Haine, Proposition 14.5.

4.6. For reduced $\mathbf{Q}$-schemes, the perfection is the seminormalisation.[26] For reduced $\mathbf{F}_p$-schemes X the perfection is the limit of X over powers of the Frobenius, as usual.

[26] STK, Tag 0EUT.

4.7 Definition. A *topological morphism* from a scheme X to a scheme Y is an morphism $\phi\colon X_{perf} \to Y$. If ϕ induces an isomorphism $X_{perf} \xrightarrow{\sim} Y_{perf}$, then it is said to be a *topological equivalence* from X to Y.

4.8. Let X and Y be schemes. Consider the following category $T(X, Y)$. The objects are diagrams

$$X \leftarrow X' \to Y$$

in which $X \leftarrow X'$ is a universal homeomorphism. A morphism

$$\text{from} \qquad X \leftarrow X' \to Y \qquad \text{to} \qquad X \leftarrow X'' \to Y$$

is a commutative diagram

$$\begin{array}{ccccc} & & X' & & \\ & \swarrow & & \searrow & \\ X & & \downarrow & & Y \\ & \nwarrow & & \nearrow & \\ & & X'' & & \end{array}$$

in which the vertical morphism is (of necessity) a universal homeomorphism. The nerve of the category $T(X, Y)$ is equivalent to the set $\mathrm{Mor}(X_{perf}, Y) \cong \mathrm{Mor}(X_{perf}, Y_{perf})$ of topological morphisms from X to Y.

4.9. The point now is that Gal, viewed as a functor from $\mathbf{Sch}_{perf}$ to categories, is *conservative*.

4.10 Definition. Let P be a property of morphisms of schemes that is stable under base change and composition. We will say that a morphism $f: X \to Y$ is *topologically* P if and only if it is topologically equivalent to a morphism of schemes $f': X' \to Y'$ with property P.

4.11. Let P be a property of morphisms of schemes that is stable under base change and composition. The class of topologically P morphisms is the smallest class of morphisms P^t that contains P and satisfies the following condition: for any commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

in which ϕ and ψ are universal homeomorphisms, the morphism f lies in P^t if and only if f' does.

A morphism $f: X \to Y$ of perfectly reduced schemes is topologically P precisely when it factors as a universal homeomorphism $X \to X'$ followed by a morphism $X' \to Y$ with property P.

4.12 Example. A morphism $f: X \to Y$ of perfectly reduced schemes is topologically radicial, surjective, universally closed, or integral if and only if it is radicial, surjective, universally closed, or integral (respectively).

4.13 Example. A morphism $f: X \to Y$ of perfectly reduced schemes is topologically étale if and only if it is étale. Indeed, if $f': X' \to Y$ is étale, then X' is perfectly reduced.[27]

[27] Rydh, B.6(ii).

5 *Finite = right fibration with finite fibres*

We've already seen that an integral morphism of schemes induces a right fibration of Galois categories and that a morphism that induces a right fibration of Galois categories must be universally closed. Let us complete this picture.

Let us begin with an obvious characterisation of quasifinite morphisms. We will say that a functor has *finite fibres* if each of its fibres is a finite set[28].

[28] which for our purposes means a finite disjoint union of contractible groupoids

5.1 Lemma. *Let $f: X \to Y$ be a morphism that is of finite type. Then f is quasifinite if and only if* $\mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ *has finite fibres.*

Since proper quasifinite morphisms are finite, Proposition 3.6 now yields:

5.2 Proposition. *Let $f: X \to Y$ be a morphism that is separated and of finite type. Then f is finite if and only if* $\mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ *is a right fibration with finite fibres.*

6 Étale = left fibration with finite fibres

6.1 Proposition. *Let $f : X \to Y$ be a morphism of schemes. If f is weakly étale, then* $\mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ *is equivalent to a left fibration. Conversely, if X and Y are perfectly reduced, if f is of finite presentation, and if* $\mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ *is a left fibration with finite fibres, then f is étale.*

Proof. Assume that f is weakly étale. Then for any geometric point $x \to X$, the morphism $X_{(x)} \to Y_{(f(x))}$ is an isomorphism, whence the functor

$$\mathrm{Gal}(X)_{x/} \simeq \mathrm{Gal}(X_{(x)}) \to \mathrm{Gal}(Y_{(f(x))}) \simeq \mathrm{Gal}(Y)_{f(x)/}$$

is an equivalence, whence $\mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ is a left fibration.

Conversely, assume that X and Y are perfectly reduced, that f is of finite presentation, and that $\mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ is a left fibration with finite fibres. So the functor $\mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ is classified by a continuous functor $\mathrm{Gal}(Y) \to \mathbf{Set}^{fin}$, which in turn corresponds to a constructible étale sheaf of finite sets on Y, which in particular coincides with the sheaf represented by X. Since the sheaf represented by X is constructible, there exists an étale map $U \to Y$ and an effective epimorphism $U \to X$ of étale sheaves on Y. By descent, $X \to Y$ is étale. □

7 Finite étale = Kan fibration with finite fibres

We may as well combine the last two entries in our dictionary.

7.1 Recollection. A *Kan fibration* is a functor that induces a Kan fibration on nerves. Equivalently, it is a functor that is both a left and right fibration. Equivalently, it is a functor $C \to D$ that is equivalent to the Grothendieck construction applied to a diagram of groupoids indexed on D^{op} that carries every morphism to an equivalence of groupoids.

7.2 Proposition. *Let $f : X \to Y$ be a morphism of perfectly reduced schemes that is separated and of finite presentation. Then f is finite étale if and only if* $\mathrm{Gal}(X) \to \mathrm{Gal}(Y)$ *is a Kan fibration with finite fibres.*

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