DESCENT PROBLEMS FOR ALGEBRAIC K-THEORY

by

Clark Barwick

We'll give you a complex, and we'll give it a name. — A. BIRD

These are notes for the "Basic Notions Seminar/Faculty Colloquium," 30 November 2009, organized by S.-T. Yau at Harvard. The standard caveats apply here: (1) These notes are very informal, and most proofs are sketched or omitted completely; even when I'm giving details, I'm skipping details. (2) Some of the ideas appear to be new, but none of the *good* ideas are mine, and all interesting results should be ascribed to others. (3) All errors are mine, and I'm duly ashamed. Really, I am.

Contents

1. The Dedekind zeta function and the Dirichlet regulator

1.1. — Suppose F a number field, with

$$[F:\mathbf{Q}] = n = r_1 + 2r_2,$$

where r_1 is the number of real embeddings, and r_2 is the number of complex embeddings. Write $\mathcal{O}_F$ for the ring of integers of F.

1.2. — Here's the power series for the *Dedekind zeta function*:

$$\zeta_F(s) = \sum_{0 \neq I \triangleleft \mathcal{O}_F} \#(\mathcal{O}_F/I)^{-s}.$$

1.3. — Here are a few key analytical facts about this power series:

(1.3.1) This power series converges absolutely for $\Re(s) > 1$.

(1.3.2) The function $\zeta_F(s)$ can be analytically continued to a meromorphic function on $\mathbf{C}$ with a simple pole at $s = 1$.

(1.3.3) There is the *Euler product expansion*:

$$\zeta_F(s) = \prod_{0 \neq p \in \operatorname{Spec} \mathcal{O}_F} \frac{1}{1 - \#(\mathcal{O}_F/p)^{-s}}.$$

(1.3.4) The Dedekind zeta function satisfies the following *functional equation*. Set

$$\xi_F(s) := \left(\frac{|\Delta_F|}{2^{2r_2}\pi^n}\right)^{s/2} \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_F(s),$$

where Δ_F is the discriminant of F (so that the volume of the fundamental domain of $\mathcal{O}_F$ in $F \otimes_{\mathbf{Q}} \mathbf{R}$ is $\sqrt{|\Delta_F|}$). Then

$$\xi_F(1-s) = \xi_F(s).$$

(1.3.5) If m is a positive integer, $\zeta_F(s)$ has a (possible) zero at $s = 1-m$ of order

$$d_m = \begin{cases} r_1 + r_2 - 1 & \text{if } m = 1; \\ r_1 + r_2 & \text{if } m > 1 \text{ is odd;} \\ r_2 & \text{if } m > 1 \text{ is even.} \end{cases}$$

Its *special value* at $s = 1-m$ is

$$\zeta_F^\star(1-m) = \lim_{s \to 1-m} (s + m - 1)^{-d_m} \zeta_F(s),$$

the first nonzero coefficient of the Taylor expansion around $1-m$.

1.4. — Our interest is in these special values of $\zeta_F(s)$ at $s = 1-m$. When $m = 1$, Dirichlet discovered an arithmetic interpretation of the special value $\zeta_F^\star(0)$, which we will briefly discuss.

1.5. — The *Dirichlet regulator map* is the logarithmic embedding

$$\rho_F^D : \mathcal{O}_F^\times / \mu_F \longrightarrow \mathbf{R}^{r_1+r_2-1},$$

where μ_F is the group of roots of unity of F. The covolume of the image lattice is the the *Dirichlet regulator* R_F^D.

Theorem 1.6 **(Dirichlet Analytic Class Number Formula).** — *The order of vanishing of $\zeta_F(s)$ at $s = 0$ is the rank $\#\mu_F$, and the special value of $\zeta_F(s)$ at $s = 0$ is given by the formula*

$$\zeta_F^\star(0) = -\frac{\#\operatorname{Pic}\mathcal{O}_F}{\#\mu_F} R_F^D.$$

1.7. — Using what we know about the lower K-theory, we have

$$K_0(\mathcal{O}_F) \cong \mathbf{Z} \oplus \operatorname{Pic}\mathcal{O}_F \quad \text{and} \quad K_1(\mathcal{O}_F) \cong \mathcal{O}_F^\times.$$

So the Dirichlet Analytic Class Number Formula reads:

$$\zeta_F^\star(0) = -\frac{\#^\tau K_0(\mathcal{O}_F)}{\#^\tau K_1(\mathcal{O}_F)} R_F^D,$$

where ${}^\tau A$ denotes the torsion subgroup of the abelian group A.

Example 1.8. — If $F = \mathbf{Q}$, the $\zeta_F(s)$ is the *Riemann zeta function* $\zeta(s)$. In this case, of course, $r_1 = 1$, and $r_2 = 0$. The Dirichlet regulator map is the map from a 0-dimensional lattice to a 0-dimensional vector space. Hence $R_{\mathbf{Q}}^D = 1$.

It follows from the functional equation that the simple pole of $\Gamma(s)$ at $s = 1$ with residue 1 gives

$$\zeta(0) = -\frac{1}{2}.$$

The Dirichlet Analytic Class Number Formula therefore encodes the observation that the class number of $\mathbf{Q}$ is 1, and $\mathbf{Q}$ contains 2 roots of unity.

Of course $\zeta(s)$ is nonzero for $s = 1-m$ if $m = 2k$ for an integer $k > 0$. In fact, the functional equation, combined with Euler's computation of $\zeta(2k)$ for positive integers k, yields:

$$\zeta(1-2k) = -\frac{B_{2k}}{2k},$$

where the B_{2k} are the *Bernoulli numbers*, given by the Taylor coefficients:

$$\frac{x}{e^x - 1} = \sum_{m \geq 0} B_m \frac{x^m}{m!}.$$

One has the recursion

$$B_m = -\sum_{\ell=0}^{m-1} \frac{1}{m+1} \binom{m+1}{\ell} B_\ell.$$

When $m = 2k+1$ for an integer $k \geq 0$, however, $\zeta(s)$ has a zero of order 1 at $s = 1-m$, and the functional equation relates the special value $\zeta^\star(1-m)$ to the value $\zeta(m)$, *viz.*:

$$\zeta^\star(-2k) = (-1)^k \frac{\pi^{2k}}{2^{2k+1}} (2k)! \zeta(2k+1).$$

There are no classical computations of $\zeta(2k+1)$ yet, though Apéry showed that $\zeta(3)$ is irrational.

What, you may ask, is the arithmetic significance of these numbers?

Example 1.9. — Suppose $F = \mathbf{Q}(\sqrt{2})$. Then $r_1 = 2$ and $r_2 = 0$. The Dirichlet regulator map is the logarithmic embedding of a one-dimensional lattice into a one-dimensional vector space, so the covolume is the logarithm of the fundamental unit: $\log(1+\sqrt{2})$.

The class number of $\mathbf{Q}(\sqrt{2})$ is 1, and $\mathbf{Q}(\sqrt{2})$ contains only 2 roots of unity, so the Dirichlet Analytic Class Number Formula gives

$$\zeta^\star_{\mathbf{Q}(\sqrt{2})}(0) = -\frac{1}{2} \log\left(1+\sqrt{2}\right).$$

In fact, this is part of a general phenomenon. If $F = \mathbf{Q}(\sqrt{\Delta})$ is a quadratic number field of discriminant Δ, then one may use the Euler product expansion to show that

$$\zeta_{\mathbf{Q}(\sqrt{\Delta})}(s) = \zeta(s) L(\chi_\Delta, s), \tag{1.9.1}$$

where $L(\chi_\Delta, s)$ is the L-function of Legendre-Kronecker character $\chi_\Delta(n) = (\Delta|n)$:

$$L(\chi_\Delta, s) := \prod_{0 \neq p \in \operatorname{Spec} \mathbf{Z}} \frac{1}{1 - (\Delta|p) p^{-s}}.$$

Thus when $\Delta = 2$, we are left with the assertion that the L-function

$$L(\chi_2, s) = \prod_{0 \neq p \in \operatorname{Spec} \mathbf{Z}} \frac{1}{1 - (-1)^{\frac{p^2-1}{8}} p^{-s}}$$

vanishes to order 1 at $s = 0$, and the special value

$$L^\star(\chi_2, 0) = \log(1+\sqrt{2}).$$

Example 1.10. — Suppose now $F = \mathbf{Q}(\sqrt{-5})$; then $r_1 = 0$, and $r_2 = 1$; so again the Dirichlet regulator is 1, and the special value of $\zeta_{\mathbf{Q}(\sqrt{-5})}(s)$ at $s = 0$ is the value. In addition, there are two roots of unity in $\mathbf{Q}(\sqrt{-5})$, and its class number is 2. Hence we are left with

$$\zeta_{\mathbf{Q}(\sqrt{-5})}(0) = -1,$$

and thus by the identity (1.9.1),

$$L(\chi_{-5}, 0) = 2.$$

2. The Borel regulator and the ur-Lichtenbaum conjecture

2.1. — Let us keep the notations from the previous section.

Theorem 2.2 **(Borel).** — *If $m > 0$ is even, then $K_m(\mathcal{O}_F)$ is finite.*

2.3. — In the early 1970s, A. Borel constructed the *Borel regulator maps*, using the structure of the homology of $\mathrm{SL}_n(\mathcal{O}_F)$. These are homomorphisms

$$\rho^B_{F,m} : K_{2m-1}(\mathcal{O}_F) \longrightarrow \mathbf{R}^{d_m},$$

one for every integer $m > 0$, generalizing the Dirichlet regulator (which is the Borel regulator when $m = 1$). Borel showed that for any integer $m > 0$ the kernel of $\rho^B_{F,m}$ is finite, and that the induced map

$$\rho^B_{F,m} \otimes \mathbf{R} : K_{2m-1}(\mathcal{O}_F) \otimes \mathbf{R} \longrightarrow \mathbf{R}^{d_m}$$

is an isomorphism. That is, the rank of $K_{2m-1}(\mathcal{O}_F)$ is equal to the order of vanishing

$$d_m = \begin{cases} r_1 + r_2 - 1 & \text{if } m = 1; \\ r_1 + r_2 & \text{if } m > 1 \text{ is odd;} \\ r_2 & \text{if } m > 1 \text{ is even.} \end{cases}$$

of the Dedekind zeta function $\zeta_F(s)$ at $s = 1 - m$. Hence the image of $\rho^B_{F,k}$ is a lattice in $\mathbf{R}^{d_k}$; its covolume is called the *Borel regulator* $R^B_{F,m}$.

Borel showed that the special value of $\zeta_F(s)$ at $s = 1 - m$ is a rational multiple of the Borel regulator $R^B_{F,m}$, *viz.*:

$$\zeta^\star_F(1-m) = Q_{F,m} R^B_{F,m}.$$

Lichtenbaum was led to give the following conjecture in around 1971, which gives a conjectural description of $Q_{F,m}$.

Conjecture 2.4 **(ur-Lichtenbaum).** — *For any integer $m > 0$,*

$$|\zeta^\star_F(1-m)| \underset{(2)}{=} \frac{\#^\tau K_{2m-2}(\mathcal{O}_F)}{\#^\tau K_{2m-1}(\mathcal{O}_F)} R^B_{F,m}.$$

(Here the notation $\underset{(2)}{=}$ indicates that one has equality up to powers of 2.)

2.5. — I have used the word "conjecture" here for historical reasons, but it seems very likely that this result is now known, and that it is the result of the Voevodsky–Rost Theorem.

Example 2.6. — Let us examine the case $F = \mathbf{Q}$. What we see is that information about ζ-values gives information about the K-theory, and information about K-groups gives information about ζ-values.

The value of the Borel regulator $R^B_{\mathbf{Q},m}$ for $m = 2k$ is 1. The ur-Lichtenbaum Conjecture thus states that

$$\frac{|B_{2k}|}{2k} \underset{(2)}{=} \frac{\#K_{4k-2}(\mathbf{Z})}{\#K_{4k-1}(\mathbf{Z})}.$$

This result is now known even more precisely: it is known that

$$\frac{|B_{2k}|}{4k} = \frac{\#K_{4k-2}(\mathbf{Z})}{\#K_{4k-1}(\mathbf{Z})},$$

and, moreover, if

$$\frac{|B_{2k}|}{4k} = \frac{c_k}{d_k}, \quad (c_k, d_k) = 1,$$

then the orders of the corresponding K-groups

$$\#K_{4k-2}(\mathbf{Z}) = \begin{cases} c_k & \text{if } k \text{ is even;} \\ 2c_k & \text{if } k \text{ is odd;} \end{cases} \quad \text{and} \quad \#K_{4k-1}(\mathbf{Z}) = \begin{cases} d_k & \text{if } k \text{ is even;} \\ 2d_k & \text{if } k \text{ is odd.} \end{cases}$$

For $m=2k+1$ for an integer $k>0$, the situation requires more care. The Borel regulator is somewhat difficult to compute. As it happens, up to a multiple of 2, $R^B_{\mathbf{Q},m}$ is the m-fold polylogarithm evaluated on a generator of $K_{2m-1}(\mathbf{Q})$. The ur-Lichtenbaum Conjecture is then the assertion that

$$|\pi^{2k}(2k)!\zeta(2k+1)| \underset{(2)}{=} \frac{\#^{\tau}K_{4k}(\mathbf{Z})}{\#^{\tau}K_{4k+1}(\mathbf{Z})} R^B_{\mathbf{Q},2k+1}.$$

Now the Voevodsky–Rost Theorem implies that for any integer $k>0$,

$$K_{4k+1}(\mathbf{Z}) = \mathbf{Z}\oplus \mathbf{Z}/2^{(k+1)\bmod 2}\mathbf{Z}.$$

Kurihara has shown that Vandiver's Conjecture is equivalent to the claim that $K_{4k}(\mathbf{Z})=0$. Given this, the ur-Lichtenbaum Conjecture becomes the claim that

$$|\zeta(2k+1)| \underset{(2)}{=} \frac{R^B_{\mathbf{Q},2k+1}}{\pi^{2k}(2k)!}.$$

The Vandiver Conjecture further implies that $K_{4k-2}(\mathbf{Z})$ and $K_{4k-1}(\mathbf{Z})$ are each cyclic. Thus the Vandiver Conjecture is equivalent to the following computation of $K_*(\mathbf{Z})$:

$$K_i(\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } i=0;\\ \mathbf{Z}/2\mathbf{Z} & \text{if } i=1;\\ \mathbf{Z}/2^{k \bmod 2}c_k\mathbf{Z} & \text{if } i=4k-2, k>0;\\ \mathbf{Z}/2^{k \bmod 2}d_k\mathbf{Z} & \text{if } i=4k-1, k>0;\\ 0 & \text{if } i=4k, k>0;\\ \mathbf{Z}\oplus\mathbf{Z}/2^{(k+1)\bmod 2}\mathbf{Z} & \text{if } i=4k+1, k>0. \end{cases}$$

3. Étale K-theory and the Quillen–Lichtenbaum conjecture

A consequence of the main conjecture of Iwasawa theory is the following.

Theorem 3.1 **(Mazur–Wiles, Wiles).** — *Suppose F a totally real number field, and suppose m even. Then*

$$|\zeta_F(1-m)| \underset{(2)}{=} \frac{\#H^2_{\text{ét}}(\mathscr{O}_F,\mathbf{Z}(m))}{\#H^0_{\text{ét}}(F,\mathbf{Q}/\mathbf{Z}(m))}$$

Theorem 3.2 **(Kolster).** — *Suppose F is an abelian number field. Then*

$$|\zeta^{\star}_F(1-m)| \underset{(2)}{=} \frac{\#H^2_{\text{ét}}(\mathscr{O}_F,\mathbf{Z}(m))}{\#H^0_{\text{ét}}(F,\mathbf{Q}/\mathbf{Z}(m))} R^B_{F,m}.$$

Results such as those above suggest that the Dedekind zeta function has to do with étale cohomology. Hence one my suspect that the ur-Lichtenbaum Conjecture has a cohomological interpretation. This is indeed true.

But let us recall the famous computation of Quillen.

Theorem 3.3 **(Quillen).** —

$$K_i(\mathbf{F}_q) \cong \begin{cases} \mathbf{Z} & \textit{if } i=0;\\ 0 & \textit{if } i=2m \textit{ and } m\geq 1;\\ \mathbf{Z}/(q^m-1)\mathbf{Z} & \textit{if } i=2m-1 \textit{ and } m\geq 1. \end{cases}$$

3.4. — For $i\geq 1$,

$$K_i(\mathbf{F}_q) \cong \begin{cases} H^2_{\text{ét}}(\mathbf{F}_q,\mathbf{Z}(m)) & \text{if } i=2m;\\ H^0_{\text{ét}}(\mathbf{F}_q,\mathbf{Q}/\mathbf{Z}(m)) & \text{if } i=2m-1. \end{cases}$$

This is compatible with the formula

$$\zeta_{\mathbf{F}_q}(s) = \frac{1}{1-q^{-s}}.$$

so that

$$\zeta_{\mathbf{F}_q}(-m) = -\frac{\# H^2_{\text{ét}}(\mathscr{O}_F, \mathbf{Z}(m))}{\# H^0_{\text{ét}}(F, \mathbf{Q}/\mathbf{Z}(m))}.$$

What, you may be tempted to ask, accounts for the shift by one here?

3.5. — The assignment $\mathscr{K} : X \longmapsto K(X)$ defines a presheaf of spectra on the category (Sch/S) of noetherian schemes of finite Krull dimension over a fixed noetherian base scheme S of finite Krull dimension. For any integer $m > 0$, one may also consider the presheaf of spectra on (Sch/S) given by mod m K-theory

$$\mathscr{K}/m : X \longmapsto K(X, \mathbf{Z}/m\mathbf{Z}).$$

We may ask whether $\mathscr{K}$ (or one of its relatives) satisfies hyperdescent with respect to various interesting topologies τ on (Sch/S). If it does, then one has a convergent descent spectral sequence

$$H^i_\tau(X, \mathscr{K}_j) \Longrightarrow K_{j-i}(X).$$

The answer depends a lot on the topology.

***Theorem 3.6* (Thomason).** — *The presheaves $\mathscr{K}$ and $\mathscr{K}/m$ satisfy Zariski — and even Nisnevich — descent on the category of noetherian S-schemes of finite Krull dimension. If the prime ℓ is invertible on S, then the Bott-inverted mod ℓ^ν K-theory $\mathscr{K}/\ell^\nu[\beta^{-1}]$ satisfies étale descent on this category. Likewise, the presheaf $\mathscr{K} \wedge H\mathbf{Q}$ satisfies étale descent.*

3.7. — Neither $\mathscr{K}$ nor $\mathscr{K}/m$ satisfies étale hyperdescent, even if m is invertible on the base scheme S. Let $\mathscr{K}^{\text{ét}}$ and $\mathscr{K}^{\text{ét}}/m$ denote the hypersheafification of these presheaves on the small étale site of S. There is, nevertheless, the following conjectural generalization of the ur-Lichtenbaum Conjecture.

***Conjecture 3.8* (Quillen–Licthenbaum).** — *Suppose m invertible on S, and suppose d the étale cohomological dimension (with $\mathbf{Z}/m\mathbf{Z}$ coefficients) of S. Then the natural morphism*

$$\mathscr{K}/m \longrightarrow \mathscr{K}^{\text{ét}}/m$$

induces isomorphisms

$$K_i(S, \mathbf{Z}/m\mathbf{Z}) \longrightarrow \mathbf{H}^{-i}_{\text{ét}}(S, \mathscr{K}^{\text{ét}}/m)$$

for $i > d$.

3.9. — Let's see how this conjecture plays out in the case of a field. First, we have the following result of Suslin.

***Theorem 3.10* (Suslin).** — *Suppose F an algebraically closed field of characteristic not ℓ. Then*

$$K(F)^\wedge_\ell \simeq ku^\wedge_\ell.$$

***Conjecture 3.11* (Quillen-Lichtenbaum, ℓ-complete version for fields).** — *Suppose F a field (not necessarily a number field), not of characteristic ℓ, of (ℓ-adic) cohomological dimension d. Suppose G_F the absolute Galois group of F. The canonical morphisms*

$$K(F) \simeq K\left(\overline{F}\right)^{\mathrm{G}_F} \longrightarrow K\left(\overline{F}\right)^{h\mathrm{G}_F}$$

have equivalent $(d+1)$-connective covers after ℓ-completion; that is, the homomorphisms

$$K_i(F)^\wedge_\ell \longrightarrow \pi_i\left(\left(K\left(\overline{F}\right)^\wedge_\ell\right)^{h\mathrm{G}_F}\right)$$

are isomorphisms for $i > d$.

3.12. — There is a well-known homotopy fixed point spectral sequence

$$H^{-s}\left(G_F, \pi_t\left(ku_\ell^\wedge\right)\right) \cong H^{-s}\left(G_F, \pi_t\left(K\left(\overline{F}\right)_\ell^\wedge\right)\right) \Longrightarrow \pi_{s+t}\left(\left(K\left(\overline{F}\right)_\ell^\wedge\right)^{hG_F}\right),$$

which therefore converges to $K_{s+t}(F)_\ell^\wedge$ for $s+t>d$ if the Quillen–Lichtenbaum Conjecture holds. One can truncate this spectral sequence, leading us to the following refinement of the Quillen–Lichtenbaum Conjecture.

***Conjecture 3.13* (Beilinson-Lichtenbaum).** — *There is a convergent spectral sequence*

$$E^2_{s,t} = \begin{cases} H^{-s}(G_F, \pi_t(ku_\ell^\wedge)) & \text{if } s+2t \geq 0; \\ 0 & \text{else,} \end{cases}$$

whose abutment is $K_{s+t}(F)_\ell^\wedge$.

3.14. — This last conjecture offers specific control over all the K-groups, but there seems to be no filtration on the spectrum $K(F)_\ell^\wedge$ yielding this spectral sequence, and no interpretation of the E^2 page as arising from the composition of two functors.

4. Equivariant stable homotopy theory and Carlsson's conjecture

4.1. — Suppose F of finite (ℓ-adic) cohomological dimension, and suppose X a geometrically connected variety over F. One may consider X with the trivial G_F action, yielding a G_F-equivariant E_∞ ring spectrum $\mathbf{K}(A_F;X)$ whose Green functor $\pi_*\mathbf{K}(A_F;X)$ assigns to any orbit (G_F/H) the K-theory of the category $\mathrm{Rep}_X[H]$ of variations of representations of H over X; in particular,

$$\pi_*^{\{1\}}\mathbf{K}(A_F;X) \cong K_*(X) \qquad \text{and} \qquad \pi_*^{G_k}\mathbf{K}(A_F;X) \cong K_*\mathrm{Rep}_X[G_F].$$

One can also use the canonical action of G_F on $\overline{X} := X \times_{\mathrm{Spec}\,F} \mathrm{Spec}\,\overline{F}$ to obtain a G_F-equivariant E_∞ ring spectrum $\mathbf{K}\left(A_F;\overline{X}\right)$ whose Green functor $\pi_*\mathbf{K}\left(A_F;\overline{X}\right)$ assigns to any orbit (G_F/H) the K-theory of $X \times_{\mathrm{Spec}\,F} \mathrm{Spec}\left(\overline{F}^H\right)$. In particular,

$$\pi_*^{\{1\}}\mathbf{K}\left(A_F;\overline{X}\right) \cong K_*\left(\overline{X}\right) \qquad \text{and} \qquad \pi_*^{G_F}\mathbf{K}\left(A_F;\overline{X}\right) \cong K_*(X).$$

Base change gives an equivariant E_∞ morphism

$$\alpha : \mathbf{K}(A_F;X) \longrightarrow \mathbf{K}\left(A_F;\overline{X}\right).$$

If ℓ is a prime with $1/\ell \in \mathscr{O}_X$, then by abuse, write $\mathbf{Z}/\ell$ for the *constant* Green functor for G_F at $\mathbf{Z}/\ell$. Now the *mod ℓ rank* yields a triangle:

$$\begin{array}{ccc} \mathbf{K}(A_F;X) & \xrightarrow{\quad\alpha\quad} & \mathbf{K}\left(A_F;\overline{X}\right) \\ & \searrow \quad \swarrow & \\ & H(\mathbf{Z}/\ell) & \end{array}$$

We can therefore form the completion (or *derived completion* in Carlsson's terminology) of both $\mathbf{K}(A_F;X)$ and $\mathbf{K}\left(A_F;\overline{X}\right)$ along $H(\mathbf{Z}/\ell)$, yielding an equivariant E_∞ morphism

$$\alpha_\ell^\wedge : \mathbf{K}(A_F;X)_\ell^\wedge \longrightarrow \mathbf{K}\left(A_F;\overline{X}\right)_\ell^\wedge.$$

Carlsson's objective is to study this morphism in the fully equivariant context, thereby eliminating the ad hoc cohomological dimension bound in the ℓ-complete Quillen–Lichtenbaum conjecture.

Theorem 4.2 **(—–, partly with Grace Lyo, conjectured for fields by Carlsson,** [?, 4.3.9]**)**
The morphism $\alpha_\ell^\wedge$ of the completions is an equivalence of G_F-equivariant E_∞ ring spectra. In particular, the G_F-fixed point spectrum $\left(\mathbf{K}\left(A_F;\overline{X}\right)_\ell^\wedge\right)^{G_F}$ coincides with the ℓ-adic completion $\mathbf{K}(X)_\ell^\wedge$, so that the G_F-fixed points of $\alpha_\ell^\wedge$ are an equivalence

$$\left(\mathbf{K}(A_F;X)_\ell^\wedge\right)^{G_F} \simeq K(X)_\ell^\wedge .$$

I will shortly turn to a description of the proof of this result. But before I do, observe that it remains to find an interpretation of the left hand side in this formula.

Conjecture 4.3. — *If F is of ℓ-adic cohomological dimension d, then the G_F-fixed point spectrum $\left(\mathbf{K}(A_F;X)_\ell^\wedge\right)^{G_F}$ and the homotopy fixed point spectrum $\left(K\left(\overline{X}\right)_\ell^\wedge\right)^{hG_F}$ have naturally equivalent $(d+1)$-connective covers.*

4.4. — It is useful to have a clear idea of what sort of objects we are dealing with. Classically, Mackey functors are additive functors indexed on a *Burnside category*, obtained by taking a group completion of a semi-additive category of spans. The ∞-categorical set-up is slightly more complicated than the classical description of the Burnside category.

Suppose G a profinite group. A G-space K will be said to be *finite* if it has finitely many components and if the isotropy subgroup is open. Denote by $B^\flat(G)^{\mathrm{fin}}$ the full subcategory of the ∞-topos $B^\flat(G)$ of G-sets spanned by the finite G-spaces.

Define the *semiexcisive Burnside ∞-category* $\mathscr{B}_G^+$ in the following manner.

(4.4.1) The objects are finite G-spaces.

(4.4.2) A morphism $K \longrightarrow M$ of finite G-spaces is a diagram

$$K \longleftarrow L \longrightarrow M$$

in $B^\flat(G)$.

(4.4.3) Given two such diagrams

$$K \longleftarrow L \longrightarrow M \quad \text{and} \quad M \longleftarrow N \longrightarrow P,$$

their composition is defined (up to a contractible choice) as the top of the pullback

$$\begin{array}{ccccc} & & L \times_M N & & \\ & \swarrow & & \searrow & \\ L & & & & N \\ \swarrow \quad \searrow & & & & \swarrow \quad \searrow \\ K \qquad\quad & & M & & \qquad\quad P. \end{array}$$

4.5. — Observe that the product $- \times -$ in $B^\flat(G)^{\mathrm{fin}}$ defines a symmetric monoidal structure on $\mathscr{B}_G^+$; note that the product of is *not* the cartesian product in $\mathscr{B}_G^+$; as a result, let us denote this symmetric monoidal structure by $\odot$.

4.6. — Note also that there are two faithful, symmetric monoidal functors

$$\ell : B^\flat(G)^{\mathrm{fin,op}} \longrightarrow \mathscr{B}_G^+ \quad \text{and} \quad r : B^\flat(G)^{\mathrm{fin}} \longrightarrow \mathscr{B}_G^+$$

that are each the identity on objects. Now a *(spectral) Mackey functor* for G is a functor $F : \mathscr{B}_G^+ \longrightarrow \mathscr{S}p$ satisfying the following properties.

(4.6.1) The functor F sends the zero object of $\mathscr{B}_G^+$ to an initial object.

(4.6.2) The functor $\ell^\star F : B^\flat(G)^{\mathrm{fin,op}} \longrightarrow D$ sends pushout squares of finite G-spaces to pushout squares in D.

(4.6.3) The functor $r^\star F : B^\flat(G)^{\mathrm{fin}} \longrightarrow D$ sends pushout squares of finite G-spaces to pushout squares in D.

The ∞-category of Mackey functors for G will be denoted $\mathscr{M}ack_G$.

4.7. — By construction, $\mathscr{M}ack_G$ is a presentable, stable ∞-category. The full subcategory $\mathscr{M}ack_{G,\geq 0}$ generated under extensions and colimits by the essential image of the functor

$$\Sigma^\infty : \mathrm{Adm}(\mathscr{B}_G^+, \mathscr{S}) \longrightarrow \mathrm{Adm}(\mathscr{B}_G^+, \mathscr{S}p) \simeq \mathscr{M}ack_G$$

defines an accessible t-structure on $\mathscr{M}ack_G$; this t-structure is both left and right complete. The heart $\mathscr{M}ack_G^\heartsuit$ of this t-structure is an abelian category of "classical" Mackey functors for the 1-truncation of G; there are corresponding functors $\pi_n : \mathscr{M}ack_G \longrightarrow \mathscr{M}ack_G^\heartsuit$.

4.8. — Given a Mackey functor A for G, one can define associated functors

$$A^\star := \ell^\star M : B^\flat(G)^{\mathrm{fin,op}} \longrightarrow \mathscr{S}p \quad \text{and} \quad A_\star := r^\star A : B^\flat(G)^{\mathrm{fin}} \longrightarrow \mathscr{S}p,$$

the first of which is contra-excisive, the second of which is excisive. This defines two "forgetful" functors

$$(-)^\star : \mathscr{M}ack_G \longrightarrow \mathrm{Exc}_{\mathrm{op}}(B^\flat(G)^{\mathrm{fin,op}}, \mathscr{S}p) \quad \text{and} \quad (-)_\star : \mathscr{M}ack_G \longrightarrow \mathrm{Exc}(B^\flat(G)^{\mathrm{fin}}, \mathscr{S}p).$$

Thus a Mackey functor for G splices together a homology theory for finite G-spaces together with a cohomology theory for finite G-spaces using a *base-change formula*; indeed, we see immediately that for any Mackey functor A for G and any pullback square

$$\begin{array}{ccccc} & & L \times_M N & & \\ & \swarrow^{f} & & \searrow^{g} & \\ L & & & & N. \\ & \searrow_{g} & & \swarrow_{f} & \\ & & M & & \end{array}$$

of $B^\flat(G)^{\mathrm{fin}}$, one must have a canonical homotopy

$$f^\star g_\star \simeq g_\star f^\star : A(L) \longrightarrow A(N).$$

4.9. — The tensor product $- \otimes -$ of Mackey functors is given by the Day convolution product, and it precisely codifies the interaction of the pullback and pushforward morphisms with the multiplicative structure that one sees in algebraic K-theory. The ∞-category $\mathscr{M}ack_G$ is *closed symmetric monoidal* with respect to the Day convolution product; consequently, there is a rich theory of A_∞ and E_∞ ring spectra in $\mathscr{M}ack_G$.

A *Green functor* is ordinarily defined as a monoid in the symmetric monoidal category of Mackey functors. But our Mackey functors are homotopical in nature; so instead we should ask for a *homotopy coherent monoid*. A *Green functor for* G is an A_∞ algebra in the symmetric monoidal category $\mathscr{M}ack_G$ of Mackey functors over S. More generally, for any operad $\mathscr{P}$, one may define a *$\mathscr{P}$-Green functor for G* simply as a $\mathscr{P}$-algebra in $\mathscr{M}ack_G$.

Now the data of a Green functor is the data of a Mackey functor A for G and a homotopy-coherently associative pairing

$$A(L) \wedge A(M) \longrightarrow A(L \odot M)$$

for every pair of finite G-spaces L and M, and a unit morphism

$$S^0 \longrightarrow A(\star).$$

There are in particular two functors attached to A, namely,

$$A^\star : B^\flat(G)^{\mathrm{fin,op}} \longrightarrow \mathscr{S}p \quad \text{and} \quad A_\star : B^\flat(G)^{\mathrm{fin}} \longrightarrow \mathscr{S}p,$$

and the homotopy associative and unital pairing on A can be viewed as two morphisms of spectra

$$A^\star(L) \wedge A^\star(M) \longrightarrow A^\star(L \odot M) \quad \text{and} \quad A_\star(L) \wedge A_\star(M) \longrightarrow A_\star(L \odot M),$$

each of which is natural in L and M.

We internalize this external tensor product by pulling back along the diagonal functor; hence for any object $K \in B^\flat(G)^{\mathrm{fin}}$, the spectrum $A(K)$ is an A_∞ algebra. The pullback functors all preserve this structure, so

$$A^\star : B^\flat(G)^{\mathrm{fin,op}} \longrightarrow \mathscr{S}p$$

can be viewed as a presheaf of A_∞ ring spectra.

On the other hand, the pushforward maps all preserve the external product, but not necessarily its internalization. It therefore follows that for any morphism $f : L \longrightarrow M$, the morphism

$$f_\star : A_\star(L) \longrightarrow A_\star(M)$$

is a morphism of $A_\star(M)$-modules.

4.10. — Let us comment on the structure of the proof, as it is relevant to what follows. Assume from now on that F is perfect, and X is smooth. (This is not strictly necessary, but it simplifies the presentation.) D. Grayson introduced a filtration on the K-theory of X:

$$\ldots \longrightarrow W^2(X) \longrightarrow W^1(X) \longrightarrow W^0(X) = K(X),$$

whose successive quotients $W^{j/j+1}(X)$ are (at least rationally) pure of weight j. This filtration is a descending sequence of (E_∞) ideals in $K(X)$. Moreover, the filtration on $K_*(X)$ given by the spectral sequence

$$E_2^{p,q} = \pi_{p+q} W^{q/q+1}(X) \Longrightarrow K_{p+q}(X)$$

coincides rationally with the γ-filtration on $K_*(X)$.

In particular, the first quotient $W^{0/1}(X)$ is $H\mathbf{Z}$, and in general, the spectra $W^{j/j+1}(X)$ are $(j+1)$-truncated, and it follows from work of Suslin that

$$\pi_{2j-i} W^{j/j+1}(X) \cong H^i_{\mathrm{mot}}(X, \mathbf{Z}(j)).$$

For our purposes here, we shall regard this left hand homotopy group as the *definition* of motivic cohomology, despite the fact that there is another "official" definition.

The key point is that: (1) this filtration can be defined equivariantly, and (2) one can use ideas of equivariant derived algebraic geometry to study the map $\alpha_\ell^\wedge$ on the various quotients.

Example 4.11. — Let us now return to the Dedekind zeta function of a number field F. In that case, there is a motivic reformulation of the Lichtenbaum conjecture:

$$|\zeta_F^\star(1-m)| \underset{(2)}{=} \frac{\#^\tau H^2_{\mathrm{mot}}(\mathscr{O}_F, \mathbf{Z}(m))}{\#^\tau H^1_{\mathrm{mot}}(\mathscr{O}_F, \mathbf{Z}(m))} R^B_{F,m}.$$

To avoid the ambiguity at 2, one should use the Beilinson regulator instead.

5. Beilinson's conjectures on special values of L-functions

5.1. — Suppose now that F is a number field and that X is a smooth proper variety of dimension n over F; denote by S its places of bad reduction. Write $\overline{X} := X \times_{\operatorname{Spec} F} \operatorname{Spec} \overline{F}$. Now for every nonzero prime $p \in \operatorname{Spec} \mathscr{O}_F$, we may choose a prime $q \in \operatorname{Spec} \mathscr{O}_{\overline{F}}$ lying over p, and we can contemplate the decomposition subgroup $D_q \subset G_F$ and the inertia subgroup $I_q \subset D_q$.

Now if ℓ is a prime over which p does not lie and $0 \leq i \leq 2n$, then the inverse ϕ_q^{-1} of the arithmetic Frobenius $\phi_q \in D_q/I_q$ acts on the I_q-invariant subspace $H^i\left(\overline{X}, \mathbf{Q}_\ell\right)^{I_q}$ of the ℓ-adic cohomology $H^i\left(\overline{X}, \mathbf{Q}_\ell\right)$. We can contemplate the characteristic polynomial of this action:

$$P_p(i,x) := \det\left(1 - x\phi_q^{-1}|_{H^i\left(\overline{X},\mathbf{Q}_\ell\right)^{I_q}}\right) \in \mathbf{Q}_\ell[x].$$

One sees that $P_p(i,x)$ does not depend on the particular choice of q.

***Conjecture 5.2* (Serre).** — *The polynomial $P_p(i,x)$ has integer coefficients that are independent of ℓ.*

5.3. — This conjecture follows from the Weil conjectures if $p \notin S$, and this is known for almost all p. We now *assume* this conjecture for all nonzero primes $p \in \operatorname{Spec} \mathscr{O}_F$. This permits us to define the *local L-factor* at the corresponding finite place $v(p)$:

$$L_{v(p)}(X,i,s) := \frac{1}{P_p(i,p^{-s})}$$

5.4. — We can also define local L-factors at infinite places. For this, we define Gamma factors

$$\Gamma_{\mathbf{R}}(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \qquad \text{and} \qquad \Gamma_{\mathbf{C}}(s) := \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s),$$

and for any infinite place v corresponding to an embedding $\sigma : F \longrightarrow \mathbf{C}$, we set

$$L_v(X,i,s) := \begin{cases} \prod_{0\leq m<i/2} \Gamma_{\mathbf{C}}(s-m)^{h^{m,i-m}} & \text{if } i \text{ is odd} \\ \Gamma_{\mathbf{R}}\left(s-\frac{i}{2}\right)^{h^+} \Gamma_{\mathbf{R}}\left(s-\frac{i}{2}+1\right)^{h^-} \prod_{0\leq m<i/2} \Gamma_{\mathbf{C}}(s-m)^{h^{m,i-m}} & \text{if } i \text{ is even,} \end{cases}$$

where $h^{m,i-m}$ is the Hodge number of $H^i\left((X \times_{\operatorname{Spec} F,\sigma} \operatorname{Spec}\mathbf{C})(\mathbf{C}),\mathbf{Q}\right)$, and h^+ and h^- are the dimensions of the $(-1)^{i/2}$ and the $-(-1)^{i/2}$ eigenspaces of $H^{i/2,i/2}$, respectively.

5.5. — With these local L-factors, we define the *L-function of X* via the Euler product expansion

$$L(X,i,s) := \prod_{0\neq p\in\operatorname{Spec}\mathcal{O}_F} L_{v(p)}(X,i,s);$$

this product converges absolutely for $\Re(s) \gg 0$. We also define the *L-function at the infinite prime*

$$L_\infty(X,i,s) := \prod_{v|\infty} L_v(X,i,s)$$

the *full L-function*

$$\Lambda(X,i,s) = L_\infty(X,i,s)L(X,i,s).$$

5.6. — Here are the expected analytical properties of the L-function of X.

(5.6.1) The Euler product converges absolutely for $\Re(s) > \frac{i}{2}+1$.

(5.6.2) $L(X,i,s)$ admits a meromorphic continuation to the complex plane, and the only possible pole occurs at $s = \frac{i}{2}+1$ for i even.

(5.6.3) $L\left(X,i,\frac{i}{2}+1\right) \neq 0$.

(5.6.4) There is a *functional equation*

$$\Lambda(X,i,s) = \varepsilon(X,i,s)\Lambda(X,i,i+1-s).$$

***Conjecture 5.7* (Beilinson).** — *Suppose $r > i/2+1$. Then the Beilinson regulator ρ induces an isomorphism*

$$\rho\otimes\mathbf{R} : H^{i+1}_{\mathrm{mot}}(X,\mathbf{Z}(r))\otimes\mathbf{R} \longrightarrow H^{i+1}_{\mathscr{D}}(X,\mathbf{R}(r)),$$

and the image of the induced homomorphism $\det H^{i+1}_{\mathrm{mot}}(X,\mathbf{Z}(r)) \longrightarrow \det H^{i+1}_{\mathscr{D}}(X,\mathbf{R}(r))$ *is equal to*

$$L^\star(X,i,i-r+1)\cdot B_{i,r},$$

where

$$B_{i,r} = \det\left(H^i_{\mathscr{B}}(X_{\mathbf{R}},\mathbf{Q}(r-1))\right)\otimes\det\left(F^r H^i_{\mathrm{dR}}(X)\right)^\vee$$

is the $\mathbf{Q}$*-structure guaranteed by Hodge theory.*

CLARK BARWICK, Department of Mathematics, Harvard University, One Oxford Street, Cambridge, MA 02138, USA
E-mail : clarkbar@gmail.com