BOREL'S COMPUTATION OF THE COHOMOLOGY OF $\mathrm{SL}(\mathscr{O}_F)$

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1. Decomposing symmetric spaces

1.1. Suppose F a number field with ring of integers $\mathscr{O}_F$. Our aim in this talk and its sequel is to compute the real cohomology of the group

$$\mathrm{SL}(\mathscr{O}_F) = \operatorname*{colim}_{n\to\infty} \mathrm{SL}_n(\mathscr{O}_F)$$

as a subgroup of

$$\mathrm{SL}(F) = \operatorname*{colim}_{n\to\infty} \mathrm{SL}_n(F)$$

This can be expressed as a limit

$$\mathrm{H}^*(\mathrm{SL}(\mathscr{O}_F), \mathbf{R}) \cong \lim_{n\to\infty} \mathrm{H}^*(\mathrm{SL}_n(\mathscr{O}_F), \mathbf{R}),$$

in the category of graded real vector spaces.

1.2. In particular, let us consider, for every integer n, the Weil restriction

$$G_n := R_{F/\mathbf{Q}} \mathrm{SL}_{n,F},$$

along with the image $\Gamma_n \subset G_n(\mathbf{Q})$ of $\mathrm{SL}_n(\mathscr{O}_F)$ under the canonical isomorphism $\mathrm{SL}_n(F) \cong G_n(\mathbf{Q})$. If S is the set of archimedean places of F (r_1 real places and r_2 complex places), then the real points of G_n are given by

$$G_n(\mathbf{R}) \cong \prod_{v\in S} G_{n,v}(\mathbf{R}),$$

where, if v is a real place, then $G_{n,v}(\mathbf{R}) \cong \mathrm{SL}_n(\mathbf{R})$, and if v is a complex place, then $G_{n,v}(\mathbf{R}) \cong \mathrm{SL}_n(\mathbf{C})$.

1.3. One sees that the symmetric space

$$X_n := G_n(\mathbf{R})/K_n$$

of maximal compact subgroups of the Lie group $G_n(\mathbf{R})$ decomposes as a product

$$X_n \cong \prod_{v\in S} X_{n,v}$$

of symmetric spaces

$$X_{n,v} := G_{n,v}(\mathbf{R})/K_{n,v}$$

of maximal compact subgroups of the Lie group $G_{n,v}(\mathbf{R})$.

Example 1.4. Suppose $F=\mathbf{Q}$. Then $r_1=1$, and $r_2=0$. The Weil restriction is trivial here, as is our decomposition of X_n:

$$X_n := \mathrm{SL}_n(\mathbf{R})/\,\mathrm{SO}_n\,.$$

This can be thought of as the space of positive-definite $n\times n$ matrices of determinant 1.

Example 1.5. Suppose $F=\mathbf{Q}(\sqrt{2})$. Then $r_1=2$, and $r_2=0$. The Weil restriction is more interesting here, and the symmetric space decomposes as

$$X_n \cong \mathrm{SL}_n(\mathbf{R})/\,\mathrm{SO}_n \times \mathrm{SL}_n(\mathbf{R})/\,\mathrm{SO}_n\,.$$

Example 1.6. Suppose $F=\mathbf{Q}(\sqrt{-5})$. Then $r_1=0$, and $r_2=1$. So here the symmetric space does not decompose:

$$X_n \cong \mathrm{SL}_n(\mathbf{C})/\,\mathrm{SU}_n$$

2. Invariant differential forms and relative Lie algebra cohomology

2.1. The key to our study will be the algebra

$$\mathrm{I}_n^* := \Omega_{X_n}^{G_n(\mathbf{R})}$$

of $G_n(\mathbf{R})$-invariant differential forms on X_n. Corresponding to our decomposition of the symmetric space above, we have a decomposition

$$\mathrm{I}_n^* \cong \bigotimes_{v\in S} \mathrm{I}_{n,v}, \qquad \text{where} \qquad \mathrm{I}_{n,v}^* := \Omega_{X_{n,v}}^{G_{n,v}(\mathbf{R})}.$$

2.2. More generally, if $\mathfrak{g}_n$ is the Lie algebra of $G_n(\mathbf{R})$, and if M is a $\mathfrak{g}$-module, then $G_n(\mathbf{R})$ acts on the space $\Omega_{X_n}(M)$ of M-valued differential forms, and we may speak of the algebra

$$\mathrm{I}_n^*(M) := \Omega_{X_n}(M)^{G_n(\mathbf{R})}$$

of $G_n(\mathbf{R})$-invariant M-valued differential forms on X_n. Suppose now that $\mathfrak{g}_{n,v}$ is the Lie algebra associated with the Lie group $G_{n,v}(\mathbf{R})$, and suppose $M_{n,v}$ a $\mathfrak{g}_{n,v}$-module. If the $\mathfrak{g}$-module M decomposes as $M\cong\bigotimes_{v\in S} M_v$, then we have a decomposition

$$\mathrm{I}_n^*(M) \cong \bigotimes_{v\in S} \mathrm{I}_{n,v}(M_v), \qquad \text{where} \qquad \mathrm{I}_{n,v}^*(M_v) := \Omega_{X_{n,v}}(M_v)^{G_{n,v}(\mathbf{R})}.$$

2.3. Now we aim to relate this algebra to the relative Lie algebra cohomology. We begin by defining a complex

$$C^*(\mathfrak{g}_{n,v};M_v) := \mathrm{Hom}(\Lambda^*\mathfrak{g}_{n,v},M_v).$$

This is a graded algebra, and it comes equipped with the *Koszul differential*

$$d\colon C^q(\mathfrak{g}_{n,v};M_v) \longrightarrow C^{q+1}(\mathfrak{g}_{n,v};M_v)$$

given by

$$\begin{aligned} df(x_0,\dots,x_q) := &\sum_i (-1)^i x_i f(x_0,\dots,\widehat{x}_i,\dots,x_q) \\ &+\sum_{i<j}(-1)^{i+j} f([x_i,x_j],x_0,\dots,\widehat{x}_i,\dots,\widehat{x}_j,\dots,x_q); \end{aligned}$$

now $d^2=0$ and $\mathrm{H}^*(\mathfrak{g}_{n,v};M_v)$ is the cohomology of this complex. This is the *Lie algebra cohomology* of $\mathfrak{g}_{n,v}$.

2.4. Now for any element $x\in\mathfrak{g}_{n,v}$, there is an endomorphism

$$\vartheta_x\colon C^q(\mathfrak{g}_{n,v};M_v) \longrightarrow C^q(\mathfrak{g}_{n,v};M_v)$$

and a linear map

$$\iota_x\colon C^q(\mathfrak{g}_{n,v};M_v) \longrightarrow C^{q-1}(\mathfrak{g}_{n,v};M_v)$$

defined by

$$\begin{aligned} \vartheta_x f(x_1,\dots,x_q) &:= x f(x_1,\dots,x_q)+\sum_i f(x_1,\dots,[x_i,x],\dots,x_q); \\ \iota_x f(x_1,\dots,x_{q-1}) &:= f(x,x_1,\dots,x_{q-1}). \end{aligned}$$

The differential above is in fact the *unique* differential d such that for any element $x\in\mathfrak{g}_{n,v}$, one has

$$\vartheta_x = d\iota_x+\iota_x d.$$

2.5. Suppose now $\mathfrak{k}_{n,v}$ the Lie algebra associated with the maximal compact $K_{n,v}$. Denote by $C^*(\mathfrak{g}_{n,v},\mathfrak{k}_{n,v};M_v)$ the subcomplex of $C^*(\mathfrak{g}_{n,v};M_v)$ consisting of elements f such that for any $x\in\mathfrak{k}_{n,v}$, one has

$$\vartheta_x f = 0 \quad\text{and}\quad \iota_x f = 0.$$

(This is compatible with the differential.) This subcomplex can be identified with the space

$$\mathrm{Hom}(\Lambda^*(\mathfrak{g}_{n,v}/\mathfrak{k}_{n,v}),M_v)^{\mathfrak{k}_{n,v}} \subset \mathrm{Hom}(\Lambda^*(\mathfrak{g}_{n,v}/\mathfrak{k}_{n,v}),M_v),$$

where the action of $\mathfrak{k}_{n,v}$ is the adjoint representation, so that the above is the subspace of elements f such that

$$\sum_i f(x_1, \dots, [x, x_i], \dots, x_q) = x f(x_1, \dots, x_q)$$

for any $x \in \mathfrak{k}_{n,v}$. The cohomology of this subcomplex is the *relative Lie algebra cohomology* $\mathrm{H}^*(\mathfrak{g}_{n,v}, \mathfrak{k}_{n,v}; M_v)$.

2.6. It is a straightforward matter to see that evaluation at $K_{n,v}e$ defines isomorphisms

$$\mathrm{I}^*_{n,v}(M_v) \cong C^*(\mathfrak{g}_{n,v}, \mathfrak{k}_{n,v}; M_v) \qquad \text{and thus} \qquad \mathrm{I}^*_n(M) \cong \bigotimes_{v \in S} C^*(\mathfrak{g}_{n,v}, \mathfrak{k}_{n,v}; M_v).$$

2.7. Now contemplate a Cartan decomposition

$$\mathfrak{g}_{n,v} = \mathfrak{k}_{n,v} \oplus \mathfrak{p}_{n,v}.$$

Then one may identify

$$C^q(\mathfrak{g}_{n,v}, \mathfrak{k}_{n,v}; M_v) \cong \mathrm{Hom}(\Lambda^q(\mathfrak{p}_{n,v}), M_v)^{\mathfrak{k}_{n,v}} \subset \mathrm{Hom}(\Lambda^q(\mathfrak{p}_{n,v}), M_v).$$

Now since $[\mathfrak{p}_{n,v}, \mathfrak{p}_{n,v}] \subset \mathfrak{k}_{n,v}$, one deduces that $d = 0$ on $C^*(\mathfrak{g}_{n,v}, \mathfrak{k}_{n,v}; M_v)$.

2.8. An immediate corollary of this is that the natural differential on $\mathrm{I}^*_{n,v}(M_v)$ — and thus on $\mathrm{I}^*_n(M)$ itself — is zero.

Example 2.9. When $F = \mathbf{Q}$, we have the usual "polar" Cartan decomposition

$$\mathfrak{sl}_n(\mathbf{R}) = \mathfrak{so}_n \oplus \mathfrak{p}_n,$$

where $\mathfrak{p}_n$ is the subspace of symmetric traceless matrices.

Example 2.10. When $F = \mathbf{Q}(\sqrt{2})$, we double the Cartan decomposition to obtain

$$\mathfrak{sl}_n(\mathbf{R}) \oplus \mathfrak{sl}_n(\mathbf{R}) = \mathfrak{so}_n \oplus \mathfrak{so}_n \oplus \mathfrak{p}_n \oplus \mathfrak{p}_n.$$

Example 2.11. When $F = \mathbf{Q}(\sqrt{-5})$, we have the Cartan decomposition

$$\mathfrak{sl}_n(\mathbf{C}) = \mathfrak{su}_n \oplus \sqrt{-1}\,\mathfrak{su}_n;$$

here $\mathfrak{su}_n$ is the subspace of traceless antihermitian matrices, and $\sqrt{-1}\,\mathfrak{su}_n$ is the subspace of traceless hermitian matrices.

3. Invariant differential forms and continuous cohomology

3.1. For any natural number q, a *continuous real* q*-cochain* on $G_{n,v}$ is a continuous function $N_q G_{n,v} \cong G^q_{n,v} \longrightarrow \mathbf{R}$. There is a natural coboundary operator, so we obtain a complex $C_c(NG_{n,v};\mathbf{R})$ and the *continuous cohomology*

$$\mathrm{H}^*_c(G_{n,v};\mathbf{R}) := \mathrm{H}^*(C_c(NG_{n,v};\mathbf{R})).$$

We wish to relate this to the algebra of invariant differential forms on $X_{n,v}$.

3.2. Suppose $(g_1,\dots,g_q) \in G^q_{n,v}$ a tuple. Then we may define a *geodesic simplex*

$$\Delta^q(g_1,\dots,g_q) \subset X_{n,v}$$

in the following manner. If $q = 1$, then we define $\Delta^1(g_1)$ as the geodesic arc connecting $\{K_{n,v}\}$ to $g_1\{K_{n,v}\}$; for $q > 1$, we define $\Delta^q(g_1,\dots,g_q)$ as the geodesic cone from $\{K_{n,v}\}$ to $g_1\Delta^{q-1}(g_2,\dots,g_q)$.

3.3. Suppose now that φ is a $G_{n,v}(\mathbf{R})$-invariant differential q-form on $X_{n,v}$. Now one may obtain a continuous real q-cochain $j(\varphi)$ on $G_{n,v}$ in the following manner. For any tuple $(g_1,\dots,g_q) \in G^q_{n,v}$, set

$$j(\varphi)(g_1,\dots,g_q) := \int_{\Delta^q(g_1,\dots,g_q)} \varphi.$$

This recipe defines a graded homomorphism

$$j : \mathrm{I}^*_{n,v} \longrightarrow C^*_c(NG_{n,v};\mathbf{R}).$$

In fact, we claim that this map j is an isomorphism, called the *van Est isomorphism*.

3.4. To begin the proof of this, let us consider the simplicial space $EG_{n,v}$ given by $E_q G_{n,v} \cong G^{q+1}_{n,v}$. The universal G-bundle can be modeled as

$$p : EG_{n,v} \longrightarrow NG_{n,v}, \qquad \text{where} \qquad p(g_0,\dots,g_q) := (g_0 g_1^{-1},\dots,g_{q-1}g_q^{-1}),$$

a smooth map of simplicial manifolds. It is a straightforward matter to show that p induces an isomorphism

$$C^*_c(NG_{n,v}) \cong C^*_c(EG_{n,v})^{G_{n,v}},$$

where the action of $G_{n,v}$ on $C^*_c(EG_{n,v})$ is given by

$$(\gamma\nu)(g_0,\dots,g_q) := \gamma\nu(g_0\gamma,\dots,g_q\gamma).$$

3.5. Now we may define a chain map

$$J\colon \Omega_{X_{n,v}} \longrightarrow C^*_c(EG_{n,v})$$

in the following manner. If $q=0$, then we define $\overline{\Delta}^0(g_0) := g_0^{-1}\{K_{n,v}\}$; for $q>0$, we define $\overline{\Delta}^q(g_0,\dots,g_q)$ as the geodesic cone from $g_0^{-1}\{K_{n,v}\}$ to $\Delta^{q-1}(g_1,\dots,g_q)$. Now define

$$J(\varphi)(g_0,\dots,g_q) := \int_{\overline{\Delta}^q(g_0,\dots,g_q)} \varphi.$$

This is a chain map by Stokes, and it is $G_{n,v}$-equivariant by the construction of the simplices $\overline{\Delta}^q(g_0,\dots,g_q)$.

3.6. Why now is this map an equivariant quasi-isomorphism? There are two possible ways of proving this. One may show, as in Hochschild–Mostow, each of $\Omega_{X_{n,v}}$ and $C^*_c(EG_{n,v})$ are continuous injective resolutions of $\mathbf{R}$ as a $G_{n,v}$-module. A perhaps simpler, or at any rate more explicit, approach is to recognize it as a special case of a simplicial de Rham theorem.

4. Relative Lie algebra cohomology and the compact twin symmetric space

4.1. On the other hand, we have the *compact twin* $X^c_{n,v}$ of $X_{n,v}$, defined in the following manner. Select a maximal compact subgroup $G^c_{n,v} \subset G_{n,v}(\mathbf{C})$ containing $K_{n,v}$, and define $X^c_{n,v} := G^c_{n,v}/K_{n,v}$. Thus if v is a real place, then

$$X^c_{n,v} = \mathrm{SU}_n/\mathrm{SO}_n,$$

and if v is a complex place, then

$$X^c_{n,v} = (\mathrm{SU}_n \times \mathrm{SU}_n)/\mathrm{SU}_n \cong \mathrm{SU}_n.$$

By an averaging argument, these compact symmetric spaces have the property that

$$\mathrm{H}^*_{\mathrm{dR}}(X^c_{n,v};\mathbf{R}) \cong \mathrm{H}^*(\Omega^{G^c_{n,v}}_{X^c_{n,v}}) \cong \mathrm{H}^*(\mathfrak{g}^c_{n,v},\mathfrak{k}_{n,v};\mathbf{R}).$$

On the other hand, the Cartan decomposition of $\mathfrak{g}^c_{n,v}$ becomes

$$\mathfrak{g}^c_{n,v} \cong \mathfrak{k}_{n,v} \oplus \sqrt{-1}\,\mathfrak{p}_{n,v},$$

so we may define an isomorphism of complexes

$$C^*(\mathfrak{g}_{n,v},\mathfrak{k}_{n,v};\mathbf{R}) \cong (\Lambda^*\mathfrak{p}^\vee_{n,v})^{\mathfrak{k}_{n,v}} \xrightarrow{\sim} (\Lambda^*(\sqrt{-1}\,\mathfrak{p}_{n,v})^\vee)^{\mathfrak{k}_{n,v}} \cong C^*(\mathfrak{g}^c_{n,v},\mathfrak{k}_{n,v};\mathbf{R})$$

by $\omega \longmapsto (\sqrt{-1})^q\omega$.

4.2. Bott periodicity thus gives computations of the relative Lie algebra cohomology stably:

$$\lim_{n\to\infty} \mathrm{H}^*(\mathfrak{g}_{n,v}, \mathfrak{k}_{n,v}) \cong \begin{cases} \mathrm{H}^*(\mathrm{SU}/\mathrm{SO}; \mathbf{R}) \cong \Lambda^*\{\sigma_i,\ \deg(\sigma_i) = 2i+1\} & \text{if } F_v \cong \mathbf{C}; \\ \mathrm{H}^*(\mathrm{SU}; \mathbf{R}) \cong \Lambda^*\{\tau_i,\ \deg(\tau_i) = 4i+1\} & \text{if } F_v \cong \mathbf{R}. \end{cases}$$

4.3. To summarize, here's a diagram:

$$\begin{array}{ccccc}
\mathrm{H}^*(\mathfrak{g}_{n,v}, \mathfrak{k}_{n,v}; \mathbf{R}) & \xrightarrow{\cong} & \mathrm{H}^*(\mathfrak{g}^c_{n,v}, \mathfrak{k}_{n,v}; \mathbf{R}) & \xrightarrow{\cong} & \mathrm{H}^*_{\mathrm{dR}}(X_{n,v}; \mathbf{R}) \\
\cong \uparrow & & & & \\
\mathrm{H}^*(\Omega_{X_{n,v}}^{G_{n,v}}) & \xrightarrow{\cong} & \mathrm{I}^*_{n,v} & & \\
\cong \downarrow & & \downarrow \cong & & \\
\mathrm{H}^*(C_c^*(EG_{n,v})^{G_{n,v}}) & \xrightarrow[\cong]{} & \mathrm{H}^*_c(G_{n,v}; \mathbf{R}) & &
\end{array}$$

In this diagram, all the objects are naturally dgas with zero differential. Bott periodicity gives an explicit stable description of $\mathrm{H}^*_c(G_{n,v}; \mathbf{R})$. Taking the tensor product of this diagram over all places $v \in S$ yields the computation

$$\lim_{n\to\infty} \mathrm{I}^*_n \cong \left(\Lambda^*\{\tau_i,\ \deg(\tau_i) = 4i+1\}\right)^{\otimes r_1} \otimes \left(\Lambda^*\{\sigma_i,\ \deg(\sigma_i) = 2i+1\}\right)^{\otimes r_2},$$

where r_1 is the number of real places of F, and r_2 is the number of complex places of F.

Example 4.4. When $F = \mathbf{Q}$, we have $r_1 = 1$, and $r_2 = 0$. So we obtain

$$\lim_{n\to\infty} \mathrm{I}^*_n \cong \Lambda^*\{\tau_i,\ \deg(\tau_i) = 4i+1\}$$

Example 4.5. When $F = \mathbf{Q}(\sqrt{2})$, we have $r_1 = 2$, and $r_2 = 0$. So we obtain

$$\lim_{n\to\infty} \mathrm{I}^*_n \cong \left(\Lambda^*\{\tau_i,\ \deg(\tau_i) = 4i+1\}\right)^{\otimes 2}$$

Example 4.6. When $F = \mathbf{Q}(\sqrt{-5})$, we have $r_1 = 0$, and $r_2 = 1$. So we obtain

$$\lim_{n\to\infty} \mathrm{I}^*_n \cong \Lambda^*\{\sigma_i,\ \deg(\sigma_i) = 2i+1\}$$

5. The real cohomology of SL($\mathcal{O}_F$)

5.1. So far our discussion has made no mention of the arithmetic subgroup $\Gamma_n \subset G_n(\mathbf{R})$. But the inclusion $\Gamma_n \hookrightarrow G_n(\mathbf{R})$ manifestly induces a map

$$j_n \colon \mathrm{H}^*(G_n(\mathbf{R}); \mathbf{R}) \longrightarrow \mathrm{H}^*(\Gamma_n; \mathbf{R})$$

Our goal is to show that if n is a natural number such that n such that $q \leq \lfloor (n-1)/4 \rfloor$, then the map j_n above induces an isomorphism

$$\mathrm{H}^q(G_n(\mathbf{R});\mathbf{R}) \cong \mathrm{H}^q(\Gamma_n, \mathbf{R}).$$

This then gives isomorphisms

$$\begin{aligned} \mathrm{H}^*(\mathrm{SL}(\mathscr{O}_F), \mathbf{R}) &\cong \bigotimes_{v\in S} \lim_{n\to\infty} \mathrm{H}^*_c(G_{n,v};\mathbf{R}) \\ &\cong \left(\Lambda^*\{\tau_i,\ \deg(\tau_i) = 4i+1\}\right)^{\otimes r_1} \otimes \left(\Lambda^*\{\sigma_i,\ \deg(\sigma_i) = 2i+1\}\right)^{\otimes r_2}. \end{aligned}$$

5.2. To understand why this map can be expected to be an isomorphism in a stable range, let us write

$$\begin{array}{ccc} \mathrm{H}^*(\mathfrak{g}_n, \mathfrak{k}_n; \mathbf{R}) & \longrightarrow & \mathrm{H}^*(\mathfrak{g}_n, \mathfrak{k}_n; \mathscr{C}^\infty(\Gamma_n \backslash G_n)) \\ \cong \uparrow & & \uparrow \cong \\ \mathrm{I}^*_n & \longrightarrow & \mathrm{I}^*_n(\mathscr{C}^\infty(\Gamma_n \backslash G_n)) \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{H}^*_c(G_n;\mathbf{R}) & \longrightarrow & \mathrm{H}^*(\Gamma_n;\mathbf{R}) \end{array}$$

We can, for example, use the idea employed in our discussion of the van Est isomorphism to reinterpret the homomorphism j_n as a map

$$j_n : \mathrm{I}^*_n \longrightarrow \mathrm{H}^*(\Gamma_n;\mathbf{R})$$

in the following manner. Suppose now that φ is a $G_n(\mathbf{R})$-invariant differential q-form on X_n. Now one may obtain a continuous real q-cochain $j_n(\varphi)$ on Γ_n in the following manner. For any tuple $(\gamma_1, \ldots, \gamma_q) \in \Gamma_n^q$, set

$$j(\varphi)(\gamma_1, \ldots, \gamma_q) := \int_{\Delta^q(\gamma_1, \ldots, \gamma_q)} \varphi.$$

6. The rank of $K_q(\mathscr{O}_F)$

6.1. As shown in the talk of I. Zakharevich, for $q \geq 2$, the rank of $K_q(\mathscr{O}_F)$ equals the dimension of the space of indecomposables in $\mathrm{H}^q(\mathrm{SL}(\mathscr{O}_F), \mathbf{R})$; hence

$$\operatorname{rk} K_q(\mathscr{O}_F) = \begin{cases} 0 & \text{if } q \equiv 0 \mod 4; \\ r_1 + r_2 & \text{if } q \equiv 1 \mod 4; \\ 0 & \text{if } q \equiv 2 \mod 4; \\ r_2 & \text{if } q \equiv 3 \mod 4; \end{cases}$$

Example 6.2. When $F = \mathbf{Q}$, we have $r_1 = 1$, and $r_2 = 0$. So we obtain

$$\operatorname{rk} K_q(\mathbf{Z}) = \begin{cases} 0 & \text{if } q \equiv 0 \mod 4; \\ 1 & \text{if } q \equiv 1 \mod 4; \\ 0 & \text{if } q \equiv 2 \mod 4; \\ 0 & \text{if } q \equiv 3 \mod 4; \end{cases}$$

Example 6.3. When $F = \mathbf{Q}(\sqrt{2})$, we have $r_1 = 2$, and $r_2 = 0$. So we obtain

$$\operatorname{rk} K_q(\mathbf{Z}[\sqrt{2}]) = \begin{cases} 0 & \text{if } q \equiv 0 \mod 4; \\ 2 & \text{if } q \equiv 1 \mod 4; \\ 0 & \text{if } q \equiv 2 \mod 4; \\ 0 & \text{if } q \equiv 3 \mod 4; \end{cases}$$

Example 6.4. When $F = \mathbf{Q}(\sqrt{-5})$, we have $r_1 = 0$, and $r_2 = 1$. So we obtain

$$\operatorname{rk} K_q(\mathbf{Z}[\sqrt{-5}]) = \begin{cases} 0 & \text{if } q \equiv 0 \mod 4; \\ 1 & \text{if } q \equiv 1 \mod 4; \\ 0 & \text{if } q \equiv 2 \mod 4; \\ 1 & \text{if } q \equiv 3 \mod 4; \end{cases}$$