$\mathscr{D}$-CRYSTALS

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Definition 1 (Grothendieck, [3, §16.8])**.** Suppose k a field, X/k a smooth variety.[1] Then the diagonal embedding $X \rightarrowtail X \times X$ is given by an ideal $I \lhd O_{X\times X}$, and the *sheaf of differential operators of order* $n \in \mathbf{N}$ is typically defined as the O_X-dual of the quotient $O_{X\times X}/I^{n+1}$:

$$D_{X/k,n} := \underline{\mathrm{Mor}}_{O_X}(O_{X\times X}/I^{n+1}, O_X).$$

The resulting filtered sheaf is a sheaf of noncommutative rings, which will simply be denoted $D_{X/k}$, and the category of right $D_{X/k}$-modules that are quasicoherent as O_X-modules will be denoted $\mathrm{Mod}^r(D_{X/k})$.[2]

Theorem 2 (Kashiwara, [6, Theorem 2.3.1])**.** *Suppose $Z \rightarrowtail X$ a closed immersion of smooth varieties over a field k of characteristic* 0*; then the category of right $D_{Z/k}$-modules is naturally equivalent to the category of right $D_{X/k}$-modules set-theoretically supported on Z.*

Theorem 3 (Hodges, [5])**.** *Suppose k algebraically closed of characteristic* 0*, and suppose X smooth over k. Then the functor $-\otimes_{O_X} D_{X/k}$ induces an equivalence of K-theory spectra*

$$K(X) \rightarrowtail K(D_{X/k}).$$

About the Proof. For affines, this follows from the K'-equivalence of a filtered ring and its 0-th filtered piece [7, Theorem 7]. The general case follows from using Kashiwara's Theorem to devise a localization sequence for $K(D_{-/k})$, which can be compared to the localization sequence for K. □

Example 4 (Bernstein-Gelfand-Gelfand, [2])**.** If X is singular, then $D_{X/k}$ is an unpleasant ring, and neither Kashiwara's nor Hodges' Theorem holds for right $D_{X/k}$-modules. To illustrate, suppose that C is the affine cone over the Fermat curve $x^3 + y^3 + z^3 = 0$ (over $\mathbf{C}$, let us say); then X is normal, and has an isolated Gorenstein singularity at the origin.

Nevertheless, the ring $D(C)$ of differential operators is neither left nor right nœtherian: if e denotes the Euler operator $x\partial_x + y\partial_y + z\partial_z$, and if $D^{(j)}(C)$ (respectively, $D_n^{(j)}(C)$) is the R-module of homogenous differential operators of degree j (resp., and of order n), then the two-sided ideals

$$J_k := \sum_{j>1} D^{(j)} + \sum_{n\geq 0} e^n D_k^{(1)}$$

form an ascending chain that does not stabilize.

5. The standard method for rectifying this is *defining deviancy down* by forcing Kashiwara's Theorem; namely, for a singular scheme Z, one embeds Z (at least locally) into a smooth scheme X and *defines* the category of right $D_{Z/k}$-modules to be the full subcategory of right $D_{X/k}$-modules set-theoretically supported along

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[1] For simplicity I will use the term "variety" for a separated nœtherian scheme of finite type.

[2] I will stick to right D-modules here.

Z. One must then show that the resulting category is invariant up to a canonical equivalence of categories.

Definition 6 (Grothendieck, [4, 4.1])**.** The *infinitesimal site* $(X_{\inf}/k)$ of X/k is the category of diagrams $X \leftarrow S \rightarrow T$ in which the morphism $S \rightarrow T$ is a closed nilimmersion of k-schemes, and the morphism $S \rightarrow X$ is étale.[3] There is a natural forgetful functor $(S,T) \mapsto T$ to the category of k-schemes; pull back the étale topology along this functor.

7. There is a stack in categories on the infinitesimal site of X:

$$\begin{aligned} \mathrm{Mod}^!_{X/k,\mathrm{qc}} : (X_{\inf}/k)^{\mathrm{op}} &\longrightarrow \mathrm{Cat} \\ (S,T) &\longmapsto \mathrm{Mod}_{\mathrm{qc}}(O_T) \\ (f,g) &\longmapsto H^0 g^! . \end{aligned}$$

Definition 8 (Beilinson-Drinfeld, [1, Definition 7.10.3])**.** A *$\mathscr{D}$-crystal* on X is a cartesian section of the stack $\mathrm{Mod}^!_{X/k,\mathrm{qc}}$. More precisely, a $\mathscr{D}$-crystal M assigns to every object (S,T) a quasicoherent O_T-module $M_{(S,T)}$ and to every morphism $(f,g) : (S,T) \rightarrow (S',T')$ an isomorphism

$$M_{(S,T)} \rightarrow H^0 g^! M_{(S',T')} .$$

The category of such will be denoted $\mathrm{Cris}^!(X/k)$.

Example 9. Suppose X a smooth k-scheme. Then for any object $(S,T) \in (X_{\inf}/k)$, let $p_T : T \rightarrow \operatorname{Spec} k$ denote the structure morphism of T, and set

$$t\omega_{X/k}(T) := H^n p_T^! \mathcal{O}_{\operatorname{Spec} k}.$$

It follows from the smoothness property of X that there exists a morphism $q : T \rightarrow X$ of k-schemes, so that $H^n p_T^! \mathcal{O}_{\operatorname{Spec} k} \cong H^0 q^! \omega_{X/k}$, where $\omega_{X/k}$ is the dualizing sheaf of top-degree differential forms.[4] Thus $t\omega_{X/k}$ is a $\mathscr{D}$-crystal.

Proposition 10 (Beilinson-Drinfeld, [1, Proposition 7.10.12])**.** *If X is a smooth k-scheme, then the category $\mathrm{Cris}^!(X/k)$ is equivalent to the category $\mathrm{Mod}^r(D_{X/k})$.*

About the Proof. The question is local, so assume X affine. If $\mathrm{pr}_1, \mathrm{pr}_2$ are the projections from the formal completion of the diagonal, $\mathrm{Cris}^!(X/k)$ is equivalent to the category of quasicoherent O_X-modules M equipped with isomorphisms $\mathrm{pr}_1^! M \cong \mathrm{pr}_2^! M$ satisfying the obvious cocycle condition. There is a natural isomorphism

$$M \otimes_{O_X} D_X \cong \mathrm{pr}_{2,\star} \mathrm{pr}_1^! M,$$

and adjunction then converts the isomorphism $\mathrm{pr}_1^! M \cong \mathrm{pr}_2^! M$ into the structure of a right D_X-module; the cocycle condition guarantees associativity. □

Theorem 11 (Beilinson-Drinfeld, [1, Lemma 7.10.11])**.** *Kashiwara's Theorem holds for $\mathscr{D}$-crystals; i.e., for any closed immersion $Z \rightarrow X$ of schemes (not necessarily smooth), the category of $\mathscr{D}$-crystals on Z is naturally equivalent to the category of $\mathscr{D}$-crystals on X set-theoretically supported on Z.*

[3] I can replace "étale" more generally with "quasi-finite" or less generally with "Zariski open immersion;" the resulting theory of $\mathscr{D}$-crystals is the same in each instance.

[4] Observe however that $\omega_{X/k}(T)$ is only a truncation of the dualizing complex $\omega_{T/k}$.

12. The appropriate functorialities of $\mathscr{D}$-crystals do not exist in general. It is more natural not to truncate $g^!$, and to consider instead the following $(\infty, 1)$-stack:

$$\begin{aligned} \mathrm{HMod}^!_{X/k,\mathrm{qc}} : \ (X_{\mathrm{inf}}/k)^{\mathrm{op}} &\longrightarrow (\infty,1)\mathrm{Cat} \\ (S,T) &\longmapsto \mathrm{Cplx}(\mathrm{Mod}_{\mathrm{qc}}(O_T)) \\ (f,g) &\longmapsto g^!. \end{aligned}$$

Definition 13. A *homotopy $\mathscr{D}$-crystal* on X is a homotopy cartesian section of the stack $\mathrm{HMod}^!_{X/k,\mathrm{qc}}$. The category of such will be denoted $\mathrm{HCris}^!(X/k)$.

Example 14. The assignment $(S,T) \mapsto \omega_{T/k}$ is a homotopy $\mathscr{D}$-crystal on X.

Proposition 15. *If X is a smooth k-scheme, then the category $\mathrm{HCris}^!(X/k)$ is equivalent to the category $\mathrm{Cplx}(\mathrm{Mod}^r(D_{X/k}))$.*

Theorem 16. *Kashiwara's Theorem holds for homotopy $\mathscr{D}$-crystals; i.e., if $Z \rightarrowtail X$ is any closed immersion of schemes (not necessarily smooth), there is a natural equivalence between the $(\infty,1)$-category of homotopy $\mathscr{D}$-crystals on Z and the full subcategory of the $(\infty,1)$-category of homotopy $\mathscr{D}$-crystals on X set-theoretically supported on X.*

Conjecture 17. *For any scheme X, the K-theory of the $(\infty,1)$-category of $\mathscr{D}$-crystals on X is naturally equivalent to $K'(X)$.*

Strategy. Again the analogue of Kashiwara's theorem permits a quick reduction to the affine case. In this case it seems possible to work directly with the definition of K-theory of $(\infty,1)$-categories, but since the definition is necessarily complicated, I have not yet managed to check all the details unless X is Cohen-Macaulay. □

References

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